9 Differential Equations

Fastrack Revision

- An equation involving an independent variable, a dependent variable and derivatives of dependent variable with respect to independent variable is known as a **differential equation.**
- **Order** of a differential equation is the order of the highest order derivative occurring in the differential equation.
- The power of highest order derivative in a differential equation is called the **degree** of that differential equation when all derivatives are free from indices and fractional powers.
- If the solution of the differential equation contains as many independent arbitrary constants as the order of the differential equation, then it is called the **general solution** or the **complete integral** of the differential equation.
- Solution obtained by giving particular values to the arbitrary constants in the general solution is called a **particular solution.**
- **Solution of Differential Equations**

(i) Variable Separable Method:

Equation of the type $\frac{dy}{dx} = F(x, y)$ can be expressed as

$$\frac{dy}{dx} = h(y) \cdot g(x)$$

$$\Rightarrow \quad \frac{1}{h(y)} dy = g(x)\, dx$$

$$\Rightarrow \quad \int \frac{1}{h(y)} dy = \int g(x)\, dx \qquad \text{(Integrating both sides)}$$

or $H(y) = G(x) + C$

where C is the integration constant.

(ii) Homogeneous Differential Equation:

An equation of the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ is solved by putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dy}{dx}$ in the given equation.

(iii) Linear Differential Equation:

(a) Solution of $\frac{dy}{dx} + Py = Q$, where P and Q are constants or functions of x only, is given by

$$y \cdot (\text{IF}) = \int (Q \times \text{IF})\, dx + C$$

where, $\text{IF} = e^{\int P\, dx}$

(b) Solution of $\frac{dx}{dy} + Px = Q$, where P and Q are constants or functions of y only, is given by

$$x \cdot (\text{IF}) = \int (Q \times \text{IF})\, dy + C$$

where, $\text{IF} = e^{\int P\, dy}$

- Equation $\frac{dy}{dx} + Py = Qy^n$ or $\frac{dx}{dy} + Px = Qx^n$, is known as Bernoulli's Equation.

This equation is converted into linear differential equation by dividing y^n or x^n respectively on both sides and after that putting $t = \frac{1}{y^{n-1}}$ or $t = \frac{1}{x^{n-1}}$ respectively.

Practice Exercise

Multiple Choice Questions

Q 1. The order of the differential equation

$$2x^2 \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0 \text{ is:}$$

a. 2
b. 1
c. 0
d. Not defined

Q 2. The order and degree (if defined) of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 = x \sin\left(\frac{dy}{dx}\right) \text{ respectively are:}$$

(CBSE 2023)

a. 2, 2
b. 1, 3
c. 2, 3
d. 2, degree not defined

Q 3. The degree of the differential equation

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2} \text{ is:}$$

(CBSE SQP 2023-24)

a. 4
b. $\frac{3}{2}$
c. 2
d. Not defined

Q 4. The order and degree of the differential equation

$$\left(1 + 3\frac{dy}{dx}\right)^2 = 4\frac{d^3y}{dx^3} \text{ respectively are:}$$

(CBSE 2023)

a. 1, $\frac{2}{3}$
b. 3, 1
c. 3, 3
d. 1, 2

Q 5. The sum of the order and degree of the differential equation $\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^3\right)$ **is:** *(CBSE 2023)*

a. 2
b. 3
c. 5
d. 0

Q 6. If m and n, respectively, are the order and degree of the differential equation $\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)\right]^4 = 0$, then $m+n=$ *(CBSE SQP 2022-23)*

a. 1 b. 2 c. 3 d. 4

Q 7. The number of arbitrary constants in the general solution of a differential equation of fourth order are: *(NCERT EXERCISE)*

a. 0 b. 2 c. 3 d. 4

Q 8. $y = x$ is a particular solution of which one of the following differential equations? *(NCERT EXERCISE)*

a. $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x$ b. $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$

c. $\frac{d^2y}{dx^2} + 1 = 0$ d. $\frac{d^2y}{dx^2} - 1 = 0$

Q 9. The general solution of the differential equation $ydx - xdy = 0$ is: *(CBSE SQP 2022-23; 2023-24)*

a. $xy = C$ b. $x = Cy^2$

c. $y = Cx$ d. $y = Cx^2$

Or

The general solution of the differential equation $\frac{y\,dx - x\,dy}{y} = 0$ is: *(NCERT EXERCISE)*

a. $xy = C$ b. $x = Cy^2$

c. $y = Cx$ d. $y = Cx^2$

Q 10. The number of solutions of the differential equation $\frac{dy}{dx} = \frac{y+1}{x-1}$, when $y(1) = 2$, is: *(CBSE 2023)*

a. zero b. one

c. two d. infinite

Q 11. The particular solution of the differential equation $\frac{dy}{dx} = e^{x+y}$, $y(0) = 0$ is:

a. $e^x + e^{-y} = 2$ b. $e^x + e^y = 2$

c. $e^{-x} + e^y = 2$ d. $e^{-x} + e^{-y} = 2$

Q 12. The solution of the differential equation $\frac{dy}{dx} = \frac{x + e^x}{y}$ is:

a. $y^2 = x^2 + 2e^x + C$ b. $x^2 = y^2 + e^x + C$

c. $x^2 = y^2 + e^{-x} + C$ d. $x^2 = y^2 + 2e^{-x} + C$

Q 13. The solution of the differential equation $\frac{dy}{dx} = \frac{1}{y + \sin y}$ is:

a. $y^2 - \sin y = 2x + C$ b. $\frac{x^2}{2} - \sin x = y + C$

c. $\frac{y^2}{2} - \cos y = x + C$ d. $\frac{x^2}{2} - \cos x = y + C$

Q 14. The solution of the differential equation $(e^x + e^{-x})\,dy - (e^x - e^{-x})\,dx = 0$ is: *(NCERT EXERCISE)*

a. $y = \log|e^x - e^{-x}| + C$ b. $y = \log|e^x + e^{-x}| + C$

c. $y = 2\log|e^x - e^{-x}| + C$ d. $y = 2\log|e^x + e^{-x}| + C$

Q 15. The solution of the differential equation $\frac{dy}{dx} = \frac{1+x^2}{1+y^2}$ is:

a. $(y-x)(y^2+x^2+xy+3) = C$

b. $(x+y)(x^2+y^2+xy+2) = C$

c. $(x+y)(x^2-y^2+2xy) = C$

d. None of the above

Q 16. Solution of the differential equation

$$\frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots} = \frac{dx - dy}{dx + dy}, \text{ is:}$$

a. $2ye^{2x} = C\cdot e^{2x} + 1$ b. $2ye^{2x} = C\cdot e^{2x} - 1$

c. $ye^{2x} = C\cdot e^{2x} + 2$ d. None of these

Q 17. Given the differential equation

$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y};\ y(1) = \pi$$

Which of the following options is correct?

a. Solution is $y^2 - \sin y = -2x^3 + C$

b. Solution is $y^2 + \sin y = 2x^3 + C$

c. $C = \pi^2 + 2\sqrt{2}$

d. $C = \pi^2 + 2$

Q 18. For the differential equation $x\frac{dy}{dx} + 2y = xy\frac{dy}{dx}$:

a. order is 1 and degree is 2

b. solution is $\ln(yx^2) = C + y$

c. order is 2 and degree is 2

d. solution is $\ln(xy^2) = C + y$

Q 19. The particular solution of $\ln\left(\frac{dy}{dx}\right) = 3x + 4y$, $y(0) = 0$, is:

a. $e^{3x} + 3e^{-4y} = 4$ b. $4e^{3x} - 3e^{-4y} = 3$

c. $3e^{3x} + 4e^{4y} = 7$ d. $4e^{3x} + 3e^{-4y} = 7$

Q 20. If $\frac{dy}{dx} = \frac{2}{x+y}$, then $x + y + 2 =$

a. Ce^y b. $Ce^{y/2}$

c. Ce^{-y} d. $Ce^{-\frac{y}{2}}$

Q 21. The solution of the differential equation $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$ is:

a. $e^x = \frac{y^3}{3} + e^y + C$ b. $e^y = \frac{x^2}{3} + e^x + C$

c. $e^y = \frac{x^3}{3} + e^x + C$ d. None of these

Q 22. The solution of the differential equation $\frac{dy}{dx} = \frac{x}{1+x^2}$ is:

a. $y = \frac{1}{2}\log|2 + x^2| + C$ b. $y = \frac{1}{2}\log(1+x) + C$

c. $y = \log(\sqrt{1+x^2}) + C$ d. None of these

Q 23. If $\frac{dy}{dx} = e^{-2y}$ and $y = 0$, when $x = 5$, then the value of x when $y = 3$ is:

a. e^5 b. $e^6 + 1$ c. $\frac{e^6 + 9}{2}$ d. $\log_e 6$

Q 24. The solution of the differential equation $\tan y \frac{dy}{dx} = \sin(x + y) + \sin(x - y)$ is:

a. $1 + 2\sin x \sin y = C \sin y$
b. $1 + 2\sin x \cos y = C \sin y$
c. $1 + 2\cos x \cdot \cos y = C \cos y$
d. None of the above

Q 25. The solution of the differential equation $y - x\frac{dy}{dx} = a\left[y^2 + \frac{dy}{dx}\right]$ is:

a. $y = C(1 - ay)(x + a)$ b. $y = C(1 + ay)(ax - 1)$
c. $y = C(y + a)(x + a)$ d. $y = C(x - a)(y - a)$

Q 26. The solution of the differential equation $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$ satisfying $y(1) = 1$, is:

a. a hyperbola b. a circle
c. $y^2 = x(1 + x) - 10$ d. $(x - 2)^2 + (y - 3)^2 = 5xy$

Q 27. If $x\,dy - y\,dx + x \cos \ln x\,dx = 0$, $y(1) = 1$, then $y(e) =$

a. $e(1 - \cos 1)$ b. $e(1 - \sin 1)$
c. $e(1 + \cos 1)$ d. $e(1 + \sin 1)$

Q 28. If $x \cos\frac{y}{x}(x\,dy + y\,dx) = y \sin\frac{y}{x}(x\,dy - y\,dx)$, then:

a. $\cos\frac{y}{x} = Cxy$ b. $\sec\frac{y}{x} = Cxy$
c. $x\cos(xy) = Cy$ d. $x\sec(xy) = Cy$

Q 29. The solution of the differential equation $\frac{x}{x^2 + y^2}dy = \left(\frac{y}{x^2 + y^2} - 1\right)dx$ is:

a. $y = x\cot(C - x)$ b. $\cos^{-1}\frac{y}{x} = (-x + C)$
c. $y = x\tan(C - x)$ d. $\frac{y^2}{x^2} = x\tan(C - x)$

Q 30. The general solution of the differential equation $y(x^2y + e^x)\,dx - e^x dy = 0$ is:

a. $x^3y - 3e^x = Cy$ b. $x^3y + 3e^x = 3Cy$
c. $y^3x - 3e^y = Cx$ d. $y^3x + 3e^y = Cx$

Q 31. If $y\,dx - x\,dy + \ln x\,dx = 0$, $y(1) = -1$, then:

a. $y + 1 + \ln x = 0$ b. $y + 1 + 2\ln x = 0$
c. $2(y + 1) + \ln x = 0$ d. $y + 1 - y\ln x = 0$

Q 32. If $(x^2 - 1)\frac{dy}{dx} + 2xy = x$, $y(0) = 0$, then $y(2) =$

a. 1 b. $\frac{1}{3}$
c. $\frac{2}{3}$ d. 2

Q 33. The solution of the differential equation $x\,dy + y\,dx = xy\,dx$, when $y(1) = 1$ is:

a. $y = \frac{e^x}{x}$ b. $y = \frac{e^x}{ex}$
c. $y = \frac{xe^x}{e}$ d. None of these

Q 34. The integrating factor for solving the differential equation $x\frac{dy}{dx} - y = 2x^2$ is: *(CBSE 2023)*

a. e^{-y} b. e^{-x} c. x d. $\frac{1}{x}$

Q 35. The integrating factor of the differential equation $x\frac{dy}{dx} + 2y = x^2$ is: *(CBSE 2020)*

a. x b. x^2 c. e^x d. e^{-x}

Q 36. The solution of the differential equation $y \cdot dx - (x + 2y^2)dy = 0$ is: *(NCERT EXERCISE; CBSE 2017)*

a. $y = 2x^2 + Cx$ b. $x = 2y^2 + Cy$
c. $y = x^2 + 2Cx$ d. $x = y^2 + 2Cy$

Q 37. The solution of the differential equation $(1 + x^2)dy + 2xy \cdot dx = \cot x\,dx$ is: *(NCERT EXERCISE)*

a. $y(1 + x^2) = \log(\cos x) + C$
b. $y(1 + x^2) = \log(\sin x) + C$
c. $y(1 - x^2) = \log(\cos x) + C$
d. None of the above

Q 38. The solution of the differential equation $\frac{dy}{dx} + (\sec x)y = \tan x, \left(0 \le x \le \frac{\pi}{2}\right)$ is: *(NCERT EXERCISE)*

a. $y(\sec x + \tan x) = (\sec x + \tan x) + C$
b. $y(\sec x - \tan x) = (\sec x - \tan x) + C$
c. $y(\sec x + \tan x) = (\sec x + \tan x) - x + C$
d. $y(\sec x - \tan x) = (\sec x - \tan x) - x + C$

Assertion & Reason Type Questions

Directions (Q. Nos. 39-45): *In the following questions, each question contains Assertion (A) and Reason (R). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are:*

a. Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)
b. Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)
c. Assertion (A) is true but Reason (R) is false
d. Assertion (A) is false but Reason (R) is true

Q 39. **Assertion (A):** The differential equation of all circles in a plane must be of order 3.

Reason (R): If three points are non-collinear, then only one circle always passing through these points.

Q 40. Assertion (A): Order of the differential equation whose solution is $y = c_1 e^{x+c_2} + c_3 e^{x+c_4}$, is 4.

Reason (R): Order of the differential equation is equal to the number of independent arbitrary constants mentioned in the general solution of the differential equation.

Q 41. Assertion (A): The solution curves of the differential equation $\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1-x^2-y^2}{x^2+y^2}}$ are circles of radius $\frac{1}{2}$.

Reason (R): The substitution $x = r\cos\theta, y = r\sin\theta$ makes the differential equation separable.

Q 42. Assertion (A): The equation of curve passing through (3, 9) which satisfies differential equation $\frac{dy}{dx} = x + \frac{1}{x^2}$ is $6xy = 3x^3 + 29x - 6$.

Reason (R): The solution of differential equation

$$\left(\frac{dy}{dx}\right)^2 - \left(\frac{dy}{dx}\right)(e^x + e^{-x}) + 1 = 0$$

is $y = e^x + c_1$ or $y = -e^{-x} + c_2$.

Q 43. Assertion (A): 'x' is not an integrating factor for the differential equation $x\frac{dy}{dx} + 2y = e^x$.

Reason (R): $x\left(x\frac{dy}{dx} + 2y\right) = \frac{d}{dx}(x^2 y)$.

Q 44. Assertion (A): $x \sin x \frac{dy}{dx} + (x + x\cos x + \sin x)y = \sin x$, $y\left(\frac{\pi}{2}\right) = 1 - \frac{2}{\pi} \Rightarrow \lim_{x \to 0} y(x) = \frac{1}{3}$.

Reason (R): The differential equation is linear with integrating factor $x(1 - \cos x)$.

Q 45. Assertion (A): If $\frac{dy}{dx} + xy = x^3 y^3$, $x > 0$, $y \geq 0$ and $y(0) = 1$, then $y(1) = \sqrt{2}$.

Reason (R): The differential equation is linear in the dependent variable $\frac{1}{y^2}$.

Answers

1. (a)	2. (d)	3. (c)	4. (b)	5. (b)	6. (c)	7. (d)	8. (b)	9. (c)	10. (a)
11. (a)	12. (a)	13. (c)	14. (b)	15. (a)	16. (b)	17. (b)	18. (b)	19. (d)	20. (b)
21. (c)	22. (c)	23. (c)	24. (c)	25. (a)	26. (a)	27. (b)	28. (b)	29. (c)	30. (b)
31. (a)	32. (c)	33. (b)	34. (d)	35. (b)	36. (b)	37. (b)	38. (c)	39. (b)	40. (d)
41. (a)	42. (b)	43. (b)	44. (a)	45. (d)					

Case Study Based Questions

Case Study 1

COVID-19 vaccine are delivered to 90 K senior citizens in a state. The rate at which COVID-19 vaccine are given is directly proportional to the number of senior citizens who have not been administered the vaccines. By the end of 3rd week, $\frac{3}{4}$th number of senior citizens have been given the COVID-19 vaccines. How many will have been given the vaccines by the end of 4th week can be estimated using the solution to the differential equation $\frac{dy}{dx} = k(90 - y)$, where x denotes the number of weeks and y the number of senior citizens who have been given the vaccines.

Based on the above information, solve the following questions:

Q 1. The order and degree of the given differential equation are:

a. 1 and 1
b. 2 and not defined
c. 1 and 0
d. 0 and 1

Q 2. Which method of solving a differential equation can be used to solve $\frac{dy}{dx} = k(90 - y)$?

a. Variable separable method
b. Solving homogeneous differential equation
c. Solving linear differential equation
d. All of the above

Q 3. The general solution of the differential equation $\frac{dy}{dx} = k(90 - y)$ is given by:

a. $\log|50 - y| = kx + C$
b. $-\log|90 - y| = kx + C$
c. $\log|90 - y| = \log|kx| + C$
d. $50 - y = kx + C$

Q 4. The value of C in the particular solution given that $y(0) = 10$ and $k = 0.025$ is:

a. log 90
b. $\log\frac{1}{80}$
c. log 80
d. 80

Q 5. Which of the following solutions may be used to find the number of senior citizens who have been given COVID-19 vaccines?

a. $y = 90 - e^{kx}$
b. $y = 90 - e^{-kx}$
c. $y = 90(1 - e^{-kx})$
d. $y = 90(e^{-kx} - 1)$

Solutions

1. Given differential equation is:

$$\frac{dy}{dx} = k(90 - y)$$

Order of equation = Order of the highest order derivative in the given differential equation = 1
and degree of equation = Degree of highest order derivative in the given differential equation = 1
So, option (a) is correct.

2. Given differential equation is:

$$\frac{dy}{dx} = k(90 - y)$$

$$\Rightarrow \quad \frac{dy}{90 - y} = k \cdot dx \quad \text{...(1)}$$

Here, we can solve the above equation by variable separable method.
So, option (a) is correct.

3. From eq. (1), $\frac{dy}{90 - y} = k\,dx$

On integrating, we get

$$\int \frac{dy}{90 - y} = k \int 1\,dx$$

$$\Rightarrow \quad -\log|90 - y| = kx + C \quad \text{...(2)}$$

which is the required general solution.
So, option (b) is correct.

4. Now, put $y(0) = 10$ and $k = 0.025$ in eq. (2), we get

$$-\log|90 - 10| = k \times 0 + C$$

$$\Rightarrow \quad C = -\log 80 = \log(80)^{-1}$$

$$\Rightarrow \quad C = \log \frac{1}{80}$$

So, option (b) is correct.

5. Let $y = 90(1 - e^{-kx})$ be the solution of the given differential equation.

$$\therefore \quad \text{LHS} = \frac{dy}{dx} = \frac{d}{dx}\{90(1 - e^{-kx})\}$$

$$= 90(0 + ke^{-kx}) = 90ke^{-kx}$$

Now, $\quad \text{RHS} = k(90 - y)$

$$= k\{90 - 90(1 - e^{-kx})\}$$

$$= k\{90 - 90 + 90e^{-kx}\}$$

$$= 90ke^{-kx} = \text{LHS}$$

Thus, $y = 90(1 - e^{-kx})$ is the required solution.
So, option (c) is correct.

Case Study 2

In a college hostel accommodating 500 students, one of the hostellers came in carrying Corona Virus and the hostel was isolated. The rate at which the virus spreads is assumed to be proportional to the product of the number of infected students and remaining students. There are 100 infected students after 5 days.

Based on the given information, solve the following questions:

Q 1. If $n(t)$ denotes the number of students infected by Corona Virus at any time t, then maximum value of $n(t)$ is:

a. 50 b. 100 c. 250 d. 500

Q 2. $\frac{dn}{dt}$ is proportional to:

a. $n(1000 - n)$ b. $n(500 - n)$
c. $n(250 - n)$ d. $n(500 + n)$

Q 3. The value of $n(5)$ is:

a. 1 b. 50 c. 100 d. 500

Q 4. The most general solution of differential equation formed in given situation is:

a. $\frac{1}{500}\log\left|\frac{500 - n}{n}\right| = \lambda t + C$

b. $\log\left|\frac{n}{100 - n}\right| = \lambda t + C$

c. $\frac{1}{500}\log\left|\frac{n}{500 - n}\right| = \lambda t + C$

d. $\log\left|\frac{100 - n}{n}\right| = \lambda t + C$

Q 5. The value of n at any time is given by:

a. $n(t) = \frac{500}{1 + 499e^{-500\lambda t}}$ b. $n(t) = \frac{500}{1 - 499e^{-500\lambda t}}$

c. $n(t) = \frac{50}{1 - 499e^{-500\lambda t}}$ d. $n(t) = \frac{50}{499 + e^{-500\lambda t}}$

Solutions

1. Since, maximum number of students in hostel is 500.
$\therefore$ Maximum value of $n(t)$ is 500.
So, option (d) is correct.

2. Let n be the number of infected students.
So, $(500 - n)$ be the remaining students.
Clearly, according to given information,

$$\frac{dn}{dt} = \lambda n(500 - n),$$

where λ is constant of proportionality.
So, option (b) is correct.

3. Since, 100 students are infected after 5 days.

$$\therefore \quad n(5) = 100$$

So, option (c) is correct.

4. We have, $\frac{dn}{dt} = \lambda n(500 - n)$

$$\Rightarrow \int \frac{dn}{n(500-n)} = \lambda \int dt$$

$$\Rightarrow \frac{1}{500}\int\left(\frac{1}{500-n}+\frac{1}{n}\right)dn = \lambda\int dt$$

$$\Rightarrow \frac{1}{500}\left[\frac{\log|500-n|}{-1}+\log|n|\right] = \lambda t + C$$

$$\Rightarrow \frac{1}{500}\log\left|\frac{n}{500-n}\right| = \lambda t + C$$

So, option (c) is correct.

5. When $t = 0$ and $n = 1$,

$$\frac{1}{500}\log\left(\frac{1}{499}\right) = C$$

$$\therefore \frac{1}{500}\left[\log\left|\frac{n}{500-n}\right| - \log\left(\frac{1}{499}\right)\right] = \lambda t$$

$$\Rightarrow \log\left|\frac{499n}{500-n}\right| = 500\lambda t$$

$$\Rightarrow \frac{499n}{500-n} = e^{500\lambda t}$$

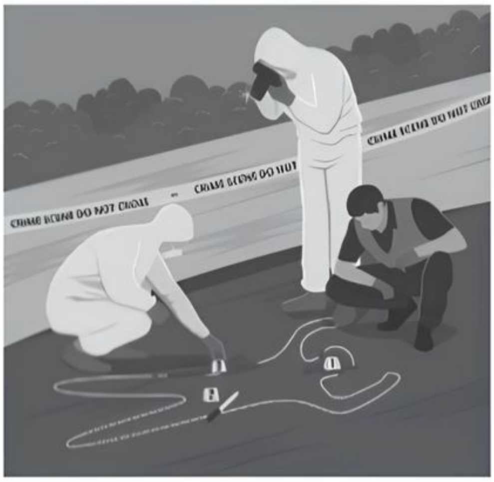
$$\Rightarrow n(t) = \frac{500e^{500\lambda t}}{499+e^{500\lambda t}} = \frac{500}{1+499e^{-500\lambda t}}$$

So, option (a) is correct.

Case Study 3

In a murder investigation, a dead body was found by a detective at exactly 9 pm. Being alert, the detective measured the body temperature and found it to be 70°F. Two hours later, the detective measured the body temperature again and found it to be 60°F, where the room temperature is 50°F. Also, it is given the body temperature at the time of death was normal, *i.e.*, 98.6°F.

Let T be the temperature of the body at any time t and initial time is taken to be 9 pm.



Based on the above information, solve the following questions:

Q 1. By Newton's law of cooling, $\frac{dT}{dt}$ is proportional to:

a. $T - 60$ b. $T - 50$ c. $T - 70$ d. $T - 98.6$

Q 2. When $t = 0$, then body temperature is equal to:

a. 50°F b. 60°F c. 70°F d. 98.6°F

Q 3. When $t = 2$, then body temperature is equal to:

a. 50°F b. 60°F c. 70°F d. 98.6°F

Q 4. The value of T at any time t is:

a. $50+20\left(\frac{1}{2}\right)^{t}$ b. $50+20\left(\frac{1}{2}\right)^{t-1}$

c. $50+20\left(\frac{1}{2}\right)^{t/2}$ d. None of these

Q 5. If it is given that $\log_e(2.43) = 0.88789$ and $\log_e(0.5) = -0.69315$, then the time at which the murder occur is:

a. 7 : 30 pm b. 6 : 30 pm

c. 6 : 00 pm d. 5 : 00 pm

Solutions

1. Given, T is the temperature of the body at any time t.
Then, by Newton's law of cooling, we get $\frac{dT}{dt} = k(T-50)$, where k is the constant of proportionality.
So, option (b) is correct.

2. From given information, we have at 9 pm, temperature is 70°F.
$\therefore$ At $t = 0$,
$T = 70°F$
So, option (c) is correct.

3. From given information, we have
At 11 pm, temperature is 60°F.
$\therefore$ At $t = 2$,
$T = 60°F$
So, option (b) is correct.

4. We have, $\frac{dT}{dt} = k(T-50) \Rightarrow \frac{dT}{T-50} = k\,dt$
On integrating both sides, we get
$$\log|T-50| = kt + \log C$$
$$\Rightarrow T - 50 = Ce^{kt}$$
Clearly, for $t = 0$, $T = 70°$
$$\Rightarrow C = 20°$$
Thus, $T - 50 = 20e^{kt}$
For $t = 2, T = 60°$
$$\Rightarrow 10 = 20e^{2k}$$
$$\Rightarrow 2k = \log\left(\frac{1}{2}\right)$$
$$\Rightarrow k = \frac{1}{2}\log\left(\frac{1}{2}\right)$$
Hence, $T = 50 + 20\left(\frac{1}{2}\right)^{\frac{t}{2}}$
So, option (c) is correct.

5. We have, $T = 50 + 20\left(\frac{1}{2}\right)^{t/2}$

Now, $98.6 = 50 + 20\left(\frac{1}{2}\right)^{\frac{t}{2}}$ $[\because T = 98.6°F]$

$$\Rightarrow \frac{48.6}{20} = \left(\frac{1}{2}\right)^{\frac{t}{2}}$$

$$\Rightarrow \quad \log\left(\frac{48.6}{20}\right) = \frac{t}{2}\log\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad \frac{t}{2} = \frac{\log\left(\frac{48.6}{20}\right)}{\log\left(\frac{1}{2}\right)}$$

$$\Rightarrow \quad t = 2\left[\frac{\log\left(\frac{48.6}{20}\right)}{\log\left(\frac{1}{2}\right)}\right]$$

$$= 2\left[\frac{\log(2.43)}{\log(0.5)}\right] = 2\left[\frac{0.88789}{-0.69315}\right]$$

$$\approx -2.56$$

So, it appears that the person was murdered 2.5 h before 9 pm *i.e.*, about 6 : 30 pm.

So, option (b) is correct.

Case Study 4

A differential equation is said to be in the variable separable form if it is expressible in the form $f(x)\,dx = g(y)\,dy$. The solution of this equation is given by $\int f(x)\,dx = \int g(y)\,dy + C$, where C is the constant of integration.

Based on the above information, solve the following questions:

Q 1. If the solution of the differential equation $\frac{dy}{dx} = \frac{ax+3}{2y+f}$ represents a circle, then the value of 'a' is:

a. 2 b. −2 c. 3 d. −4

Q 2. The differential equation $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$ determines a family of circle with:

a. variable radii and fixed centre (0, 1)

b. variable radii and fixed centre (0, −1)

c. fixed radius 1 and variable centre on X-axis

d. fixed radius 1 and variable centre on Y-axis

Q 3. If $\frac{dy}{dx} = y + 1$, $y(0) = 1$, then $y(\ln 2)$ is equal to:

a. 1 b. 2

c. 3 d. 4

Q 4. The solution of the differential equation $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ is:

a. $e^x = \frac{y^3}{3} + e^y + C$ b. $e^y = \frac{x^2}{3} + e^x + C$

c. $e^y = \frac{x^3}{3} + e^x + C$ d. None of these

Q 5. If $\frac{dy}{dx} = y\sin 2x$, $y(0) = 1$, then its solution is:

a. $y = e^{\sin^2 x}$ b. $y = \sin^2 x$

c. $y = \cos^2 x$ d. $y = e^{\cos^2 x}$

Solutions

1. We have, $\frac{dy}{dx} = \frac{ax+3}{2y+f}$

$\Rightarrow \quad (ax+3)\,dx = (2y+f)\,dy$

$\Rightarrow \quad a\frac{x^2}{2} + 3x = y^2 + fy + C$ [on integrating]

$\Rightarrow \quad -\frac{a}{2}x^2 + y^2 - 3x + fy + C = 0$

This will represent a circle, if $\frac{-a}{2} = 1 \Rightarrow a = -2$

[$\because$ in circle, coefficient of x^2 = coefficient of y^2]

So, option (b) is correct.

2. We have, $\frac{y\,dy}{\sqrt{1-y^2}} = dx$

On integration, we get $-\sqrt{1-y^2} = x + C$

$\Rightarrow \quad 1 - y^2 = (x+C)^2$

$\Rightarrow \quad (x+C)^2 + y^2 = 1,$

which represents a circle with radius 1 and centre on the X-axis.

So, option (c) is correct.

3. $y' = y + 1 \Rightarrow \frac{dy}{y+1} = dx$

$\Rightarrow \quad \ln|y+1| = x + C$ [on integrating]

Now, $y(0) = 1 \Rightarrow C = \ln 2$

$\therefore \quad \ln\left|\frac{y+1}{2}\right| = x \Rightarrow y + 1 = 2e^x$

So, $y(\ln 2) = -1 + 2e^{\ln 2} = -1 + 4 = 3$

So, option (c) is correct.

4. From the given differential equation, we have

$$\frac{dy}{dx} = \frac{e^x + x^2}{e^y}$$

$\Rightarrow \quad e^y dy = (e^x + x^2)\,dx$

On integrating, we get $e^y = e^x + \frac{x^3}{3} + C$

So, option (c) is correct.

5. We have, $\frac{dy}{dx} = y\sin 2x$

$\Rightarrow \quad \int\frac{dy}{y} = \int \sin 2x\,dx \Rightarrow \log|y| = -\frac{\cos 2x}{2} + C$

Since $x = 0$, $y = 1$, therefore $C = 1/2$

Now, $\log|y| = \frac{1}{2}(1 - \cos 2x)$

$\Rightarrow \quad \log|y| = \sin^2 x \Rightarrow y = e^{\sin^2 x}$

So, option (a) is correct.

Case Study 5

A thermometer reading 0°F is taken outside. Ten minutes later, the thermometer reads 70°F. After another 5 min, the thermometer reads 50°F. At any time t, the thermometer reading be T°F and the outside temperature be S°F.

Based on the given information, solve the following questions:

Q 1. If λ is positive constant of proportionality, then find $\frac{dT}{dt}$.

Q 2. Find the general solution of differential equation formed in given situation.

Q 3. Find the value of constant of integration C in the solution of differential equation formed in given situation.

Or

15 minutes later, the thermometer reads 50°F, find the outside temperature.

Solutions

1. Given, at any time t, the thermometer reading be T°F and the outside temperature be S°F.

Then, by Newton's law of cooling, we have

$$\frac{dT}{dt} \propto (T-S) \Rightarrow \frac{dT}{dt} = -\lambda(T-S)$$

2. We have, $\frac{dT}{dt} = -\lambda(T-S)$

$$\Rightarrow \frac{dT}{T-S} = -\lambda\, dt$$

$$\Rightarrow \int \frac{1}{T-S} dT = -\lambda \int dt$$

$$\Rightarrow \log|T-S| = -\lambda t + C$$

3. Since, at $t=0$, $T=100$°F

$\therefore \log|100-S| = 0 + C$

$\Rightarrow C = \log|100-S|$

Or

Since, at $t=15$, $T=50$°F

$\therefore \log|50-S| = -\lambda \times 15 + \log|100-S|$

$$\Rightarrow 15\lambda = \log\left|\frac{100-S}{50-S}\right|$$

$$\Rightarrow e^{15\lambda} = \frac{100-S}{50-S} \Rightarrow S = 50\left(\frac{2-e^{15\lambda}}{1-e^{15\lambda}}\right)$$

Case Study 6

It is known that, if the interest is compounded continuously, the principal changes at the rate equal to the product of the rate of bank interest per annum and the principal. Let P denotes the principal at any time t and rate of interest be r% per annum.

Based on the given information, solve the following questions:

Q 1. If P_0 is the initial principal, then find the solution of differential equation formed in given situation.

Q 2. If the interest is compounded continuously at 5% per annum, in how many years will ₹ 100 double itself?

Or

At what interest rate will ₹ 100 double itself in 10 yr? ($\log_e 2 = 0.6931$)

Q 3. How much will ₹ 1000 be worth at 5% interest after 10 yr? ($e^{0.5} = 1.648$)

Solutions

1. Here, P denotes the principal at any time t and the rate of interest be r% per annum compounded continuously, then according to the law given in the problem, we get

$$\frac{dP}{dt} = \frac{Pr}{100}$$

$$\Rightarrow \frac{dP}{P} = \frac{r}{100} dt \Rightarrow \int \frac{1}{P} dP = \frac{r}{100}\int dt$$

$$\Rightarrow \log P = \frac{rt}{100} + C \quad \text{...(1)}$$

At $t=0$, $P=P_0$

$\therefore C = \log P_0$

So, $\log P = \frac{rt}{100} + \log P_0$

$$\Rightarrow \log\left(\frac{P}{P_0}\right) = \frac{rt}{100} \quad \text{...(2)}$$

2. We have, $r = 5\%$, $P_0 = ₹\,100$ and $P = ₹\,200 = 2P_0$

Substituting these values in eq. (2), we get

$$\log 2 = \frac{5}{100} t$$

$$\Rightarrow t = 20 \log_e 2 = 20 \times 0.6931 \text{ yr} = 13.862 \text{ yr}$$

Or

We have,

$P_0 = ₹\,100$, $P = ₹\,200 = 2P_0$ and $t = 10$ yr

Substituting these values in eq. (2), we get

$$\log 2 = \frac{10r}{100}$$

$$\Rightarrow r = 10 \log 2 = 10 \times 0.6931 = 6.931$$

3. We have, $P_0 = ₹\,1000$, $r = 5\%$ and $t = 10$

Substituting these values in eq. (2), we get

$$\log\left(\frac{P}{1000}\right) = \frac{5 \times 10}{100} = \frac{1}{2} = 0.5$$

$$\Rightarrow \frac{P}{1000} = e^{0.5} \Rightarrow P = 1000 \times 1.648 = ₹\,1648$$

Case Study 7

An equation involving derivatives of the dependent variable with respect to the independent variables is called a differential equation. A differential equation of the form $\frac{dy}{dx} = F(x, y)$ is said to be homogeneous, if $F(x, y)$ is a homogeneous function of degree zero, whereas a function $F(x, y)$ is a homogeneous function of degree n, if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$. To solve a homogeneous differential equation of the type $\frac{dy}{dx} = F(x, y) = g\left(\frac{y}{x}\right)$, we make the substitution $y = vx$ and then separate the variables.

Based on the above information, solve the following questions: (CBSE 2023)

Q 1. Show that $(x^2 - y^2)dx + 2xy\,dy = 0$ is a differential equation of the type $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$.

Q 2. Solve the above equation to find its general solution.

Solutions

1. We have,

$$(x^2 - y^2)dx + 2xy\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = F(x, y) \quad \text{(say)}$$

Now, $F(\lambda x, \lambda y) = \frac{\lambda^2 y^2 - \lambda^2 x^2}{2\lambda x \lambda y} = \frac{\lambda^2 (y^2 - x^2)}{\lambda^2 2xy}$

$$= \lambda^0 \frac{(y^2 - x^2)}{2xy} = \lambda^0 \left\{ \frac{\left(\frac{y}{x}\right)^2 - 1}{2\left(\frac{y}{x}\right)} \right\}$$

Here, degree of function is 0, so given differential equation is a homogeneous equation.

Hence, the given differential equation of the type $g\left(\frac{y}{x}\right)$.

2. $\because \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad v + \frac{xdv}{dx} = \frac{v^2x^2 - x^2}{2xvx}$$

$$\Rightarrow \quad \frac{xdv}{dx} = \frac{v^2 - 1}{2v} - v$$

$$\Rightarrow \quad \frac{xdv}{dx} = \frac{-v^2 - 1}{2v}$$

$$\Rightarrow \quad \frac{-2v}{v^2 + 1}dv = \frac{dx}{x}$$

Integrating both sides, we get

$$-\int \frac{2v}{v^2 + 1}dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad -\log|v^2 + 1| = \log|x| + \log C$$

$$\Rightarrow \quad \log\left|\frac{1}{v^2 + 1}\right| = \log|x|C$$

$$\Rightarrow \quad \frac{1}{v^2 + 1} = xC$$

$$\Rightarrow \quad \frac{1}{\frac{y^2}{x^2} + 1} = xC \quad \left[\text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \frac{x^2}{y^2 + x^2} = xC \Rightarrow x = (x^2 + y^2)C$$

Case Study 8

A first order and first degree differential equation in which the degree of dependent variable and its derivative is one and they do not get multiplied together, is called a linear differential equation.

Consider the given equation $\frac{dy}{dx} + Py = Q$. This equation is known as linear differential equation. Here integrating factor *i.e.*, $\text{IF} = e^{\int P\,dx}$ and the complete solution is given by

$$y(\text{IF}) = \int Q \times (\text{IF})\,dx + C,$$

where P and Q are constants or some function of x.

Now consider the given equation

$$dy = \cos x\,(2 - y \operatorname{cosec} x)\,dx$$

Based on the above information, solve the following questions:

Q 1. Find the value of integrating factor (IF).

Q 2. Find the general solution of the given equation.

Q 3. If $y\left(\frac{\pi}{2}\right) = 2$, then find the particular solution of given equation.

Or

If $x = \frac{\pi}{6}$, then find the value of y.

Solutions

1. Given differential equation can be written in linear differential equation form

$$\frac{dy}{dx} = \cos x\,(2 - y \operatorname{cosec} x)$$

$$\Rightarrow \quad \frac{dy}{dx} = 2\cos x - \cos x \cdot \operatorname{cosec} x \cdot y$$

$$\Rightarrow \quad \frac{dy}{dx} + \cos x \cdot \frac{1}{\sin x} y = 2\cos x$$

$$\Rightarrow \quad \frac{dy}{dx} + \cot x \cdot y = 2\cos x$$

On comparing with $\frac{dy}{dx} + Py = Q$, we get

$$P = \cot x \text{ and } Q = 2\cos x$$

Now, $\text{IF} = e^{\int P\,dx} = e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$

2. Complete solution is

$$y \cdot \text{IF} = \int Q \cdot \text{IF}\,dx + C$$

$$\Rightarrow y \cdot \sin x = \int 2\cos x \cdot \sin x\,dx + C$$

$$\Rightarrow y \cdot \sin x = \int \sin 2x\,dx + C \Rightarrow y \cdot \sin x = -\frac{1}{2}\cos 2x + C$$

which is the required general solution.

3. Now, put $x = \frac{\pi}{2}$ and $y = 2$ in general solution, we get

$$2 \cdot \sin\frac{\pi}{2} = -\frac{1}{2}\cos 2 \cdot \frac{\pi}{2} + C \Rightarrow 2 \times 1 = -\frac{1}{2}\cos \pi + C$$

$$\Rightarrow 2 = -\frac{1}{2}(-1) + C \Rightarrow C = 2 - \frac{1}{2} = \frac{3}{2}$$

$\therefore$ Required particular solution is

$$y \cdot \sin x = -\frac{1}{2}\cos 2x + \frac{3}{2}$$

$$\Rightarrow \quad 2y \sin x = -\cos 2x + 3$$

Or

Now, put $x = \frac{\pi}{6}$ in the particular solution, we get

$$2y\sin\frac{\pi}{6} = -\cos 2\cdot\frac{\pi}{6} + 3$$

$$\Rightarrow \quad 2y \times \frac{1}{2} = -\cos\frac{\pi}{3} + 3$$

$$\Rightarrow \quad y = -\frac{1}{2} + 3 = \frac{5}{2}$$

Very Short Answer Type Questions

Q 1. Find the order and the degree of the differential equation $x^2\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^4$. *(CBSE 2019)*

Q 2. Write the order and the degree of the following differential equation

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0.$$ *(CBSE 2019)*

Q 3. Write the sum of the order and degree of the following differential equation:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 5$$ *(CBSE SQP 2022 Term-2)*

Q 4. Write the degree of the following differential equations:

(i) $\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 2x^2\log\left(\frac{d^2y}{dx^2}\right)$

(NCERT EXEMPLAR; CBSE 2019)

(ii) $\frac{d^3y}{dx^3} + y^2 + e^{(dy/dx)} = 0$ *(NCERT EXERCISE)*

Q 5. Write the sum of the order and degree of the differential equation $1 + \left(\frac{dy}{dx}\right)^4 = 7\left(\frac{d^2y}{dx^2}\right)^3$.

Q 6. How many arbitrary constants are there in the particular solution of the differential equation

$$\frac{dy}{dx} = -4xy^2; y(0) = 1?$$ *(CBSE 2020)*

Q 7. Show that the function $y = ax + 2a^2$ is a solution of the differential equation $2\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) - y = 0$.

(CBSE 2020)

Q 8. Write the solution of the differential equation $\frac{dy}{dx} = 2^{-y}$.

Q 9. Find the general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$. *(CBSE 2020, 19)*

Q 10. Find the solution of the differential equation

$$\frac{dy}{dx} = x^3 \cdot e^{-2y}.$$

Q 11. Solve the differential equation $\frac{dy}{dx} = \frac{1+x^2}{y + \sin y}$.

Q 12. Solve the differential equation:

$$(1-x)\,dy - (3+y)\,dx = 0$$

Q 13. For what value of n is the following a homogeneous differential equation

$$\frac{dy}{dx} = \frac{x^3 - y^n}{x^2y + xy^2}?$$ *(CBSE 2020)*

Q 14. Write the integrating factor of the differential equation $\sqrt{x}\frac{dy}{dx} + y = e^{-2\sqrt{x}}$.

Q 15. Write the integrating factor of the following differential equation:

$$(1+y^2) + (2xy - \cot y)\frac{dy}{dx} = 0.$$

Short Answer Type-I Questions

Q 1. Verify that $ax^2 + by^2 = 1$ is a solution of differential equation $x(y y_2 + y_1^2) = yy_1$.

(NCERT EXEMPLAR; CBSE 2017)

Q 2. Solve the differential equation $\frac{dy}{dx} + \left(\frac{1+y^2}{x}\right) = 0$.

Q 3. Solve the differential equation $\frac{dy}{dx} = xy + x + y + 1$.

Q 4. Solve $\log\frac{dy}{dx} = ax + by$.

(NCERT EXERCISE; CBSE 2022 Term-2)

Q 5. Solve the differential equation

$$\log\left(\frac{dy}{dx}\right) = x - y.$$ *(CBSE 2022 Term-2)*

Q 6. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}$. *(CBSE 2022 Term-2)*

Q 7. Solve the differential equation:

$$\sec x \cdot \operatorname{cosec} y\,dx + \sec y\,dy = 0$$

Q 8. Solve the differential equation:

$$(\cos^2 x - \sin^2 x)\,dy + 2\sin y\cos y\,dx = 0$$

Q 9. Solve the differential equation $\frac{dy}{dx} + \frac{1+y^2}{y} = 0$.

Q 10. Solve $[(1+e^x)\,y]\,dy = [(y+1)\,e^x]\,dx$.

Q 11. Solve the differential equation:

$$(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0$$

Q 12. Solve the differential equation:

$$\sec^2 x\tan y\,dx + \sec^2 y\tan x\,dy = 0$$

(NCERT EXERCISE)

Q 13. Solve the differential equation $\frac{d^2y}{dx^2} = \sin x$.

Q 14. Solve the following differential equation

$\frac{dy}{dx} = x^3 \operatorname{cosec} y$, given that $y(0) = 0$ *(CBSE 2020)*

Q 15. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$; given that $y(0) = \sqrt{3}$. *(NCERT EXERCISE)*

Q 16. Find the general solution of the differential equation $\frac{dy}{dx} + \frac{2}{x}y = x$. *(NCERT EXERCISE)*

Q 17. Find the general solution of the differential equation $\frac{dy}{dx} + 2y = e^{3x}$.

Q 18. Find the particular solution of the differential equation $x\frac{dy}{dx} - y = x^2 \cdot e^x$, given $y(1) = 0$. *(CBSE 2022 Term-2)*

Short Answer Type-II Questions

Q 1. Solve the differential equation

$\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$. *(NCERT EXEMPLAR)*

Q 2. Solve the differential equation

$(x+1)\frac{dy}{dx} = 2e^{-y} - 1; y(0) = 0$. *(CBSE 2019)*

Q 3. Find the general solution of the differential equation $e^x \tan y\, dx + (1-e^x)\sec^2 y\, dy = 0$. *(CBSE 2023)*

Q 4. Find the equation of that curve which passes through the point $(-2, 3)$ and whose slope of tangent at any point (x, y) is $\frac{2x}{y^2}$. *(NCERT EXERCISE)*

Q 5. Solve the differential equation:

$x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$ *(CBSE SQP 2022-23)*

Q 6. Find the general solution of the differential equation $x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$. *(CBSE SQP 2022 Term-2)*

Q 7. Find the general solution of the differential equation $x\frac{dy}{dx} = y(\log y - \log x + 1)$. *(CBSE 2022 Term-2)*

Q 8. Show that the family of curves for which $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$, is given by $x^2 - y^2 = Cx$. *(NCERT EXERCISE; CBSE 2017)*

Q 9. Solve the differential equation:

$(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$

Q 10. Find the general solution of the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$. *(NCERT EXERCISE; CBSE 2017)*

Q 11. Solve the differential equation:

$ye^{x/y}dx = (xe^{x/y} + y^2)dy,\ (y \neq 0)$. *(CBSE SQP 2023-24)*

Or

Find the general solution of the differential equation $ye^{x/y}\,dx = (xe^{x/y} + y^2)\,dy, y \neq 0$. *(NCERT EXERCISE; CBSE 2020)*

Q 12. Find the particular solution of the differential equation $2y\,e^{x/y}\,dx + (y - 2x\,e^{x/y})\,dy = 0$, given that $x = 0$, when $y = 1$. *(NCERT EXERCISE; CBSE 2017)*

Q 13. Solve the differential equation:

$x\sin\left(\frac{y}{x}\right)\frac{dy}{dx} + x - y\sin\left(\frac{y}{x}\right) = 0$, given that $x = 1$, when $y = \frac{\pi}{2}$. *(CBSE 2020)*

Q 14. Solve the differential equation:

$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$. *(NCERT EXEMPLAR)*

Q 15. Solve the differential equation:

$\frac{dy}{dx} - \frac{2x}{1+x^2}y = x^2 + 2$. *(CBSE 2019)*

Q 16. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$, $y(1) = 0$. *(CBSE 2023)*

Q 17. Solve the differential equation:

$(\tan^{-1} y - x)\,dy = (1 + y^2)\,dx$. *(NCERT EXERCISE; CBSE 2017)*

Q 18. Solve the differential equation:

$ydx + (x - y^2)dy = 0$. *(CBSE SQP 2022-23)*

Q 19. Find the general solution of the differential equation $\frac{dy}{dx} - y = \sin x$. *(CBSE 2017)*

Q 20. Solve the differential equation:

$(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$, subject to the initial condition $y(0) = 0$. *(NCERT EXEMPLAR, CBSE 2019)*

Q 21. Find the particular solution of the differential equation $\frac{dy}{dx} + 2y\tan x = \sin x$, given that $y = 0$ when $x = \frac{\pi}{3}$. *(NCERT EXERCISE; CBSE 2018)*

Q 22. Find the particular solution of the differential equation $\frac{dy}{dx} + y\cot x = 2x + x^2\cot x$, $x \neq 0$, $y\left(\frac{\pi}{2}\right) = 0$. *(NCERT EXERCISE; CBSE 2017)*

Q 23. Find the particular solution of the differential equation $x\frac{dy}{dx}+x\cos^2\left(\frac{y}{x}\right)=y$; given that when $x=1, y=\frac{\pi}{4}$. *(CBSE 2022 Term-2)*

Q 24. Solve the differential equation:

$$(\cos^2 x)\frac{dy}{dx}+y=\tan x;\left(0\le x<\frac{x}{2}\right).$$

(CBSE SQP 2023-24)

Long Answer Type Questions

Q 1. Solve the following differential equation

$$\sqrt{1+x^2+y^2+x^2y^2}+xy\frac{dy}{dx}=0.$$ *(CBSE 2015)*

Q 2. Solve the differential equation $\frac{d^2y}{dx^2}=x\sin x+e^x$.

Q 3. Find the equation of that curve which passes through the point (0, 0) and having differential equation $y'=e^x\sin x$. *(NCERT EXERCISE)*

Q 4. Solve the differential equation:

$$x^2y\,dx-(x^3+y^3)\,dy=0$$

Q 5. Solve the differential equation:

$$(x^2-y^2)\,dx+2xy\,dy=0$$ *(NCERT EXERCISE)*

Q 6. Prove that $x^2-y^2=C(x^2+y^2)^2$ is the general solution of the differential equation $(x^3-3xy^2)\,dx=(y^3-3x^2y)\,dy$, where C is a parameter. *(NCERT EXERCISE; CBSE 2017)*

Q 7. Solve the differential equation:

$$(x\,dy-y\,dx)\,y\sin\left(\frac{y}{x}\right)=(y\,dx+x\,dy)\,x\cos\left(\frac{y}{x}\right)$$

(NCERT EXERCISE)

Q 8. Solve the differential equation:

$$x^2dy+(xy+y^2)\,dx=0, y(1)=1$$ *(NCERT EXERCISE)*

Q 9. Find the particular solution of the differential equation $(x-y)\frac{dy}{dx}=(x+2y)$, given that $y=0$, when $x=1$. *(NCERT EXERCISE; CBSE 2017)*

Q 10. Solve the differential equation:

$x\,dy-y\,dx=\sqrt{x^2+y^2}\,dx$, given that $y=0$, when $x=1$. *(NCERT EXERCISE; NCERT EXEMPLAR; CBSE 2019)*

Q 11. Solve $ye^y dx=(y^3+2xe^y)\,dy, y(0)=1$.

Q 12. Solve $x\frac{dy}{dx}+y=x^3y^6$.

Q 13. Solve $\frac{dy}{dx}+y\tan x=y^2\sec x$.

Q 14. Find the particular solution of the differential equation $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$, given that $y=0$ when $x=1$.

(NCERT EXERCISE; NCERT EXEMPLAR; CBSE 2017)

Q 15. Find the particular solution of the differential equation $\frac{dy}{dx}+y\cot x=\frac{2}{1+\sin x}$ given that $y=0$, when $x=\frac{\pi}{4}$. *(CBSE SQP 2022 Term-2)*

Q 16. Solve the differential equation:

$\frac{dy}{dx}-3y\cot x=\sin 2x$, when at $x=\frac{\pi}{2}$, $y=2$, find the particular solution. *(NCERT EXERCISE; CBSE 2017)*

Q 17. Solve the differential equation:

$x\frac{dy}{dx}+y=x\cos x+\sin x$, given that $y=1$ when $x=\frac{\pi}{2}$. *(CBSE 2017)*

Solutions

Very Short Answer Type Questions

1. Given differential equation is $x^2\frac{d^2y}{dx^2}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$.

TiPS

- *Degree of the differential equation should be free from radical sign and fractions.*
- *Order and degree (if defined) of a differential equation are always positive integers.*

(i) Order of differential equation = Order of the highest order derivative in the given differential equation = 2

(ii) Degree of differential equation = Power of highest order derivative in the given differential equation = 1.

COMMON ERROR

Some students write the degree as 4 or 8 considering it as the highest power.

2. Given differential equation is

$$x^3\left(\frac{d^2y}{dx^2}\right)^2+x\left(\frac{dy}{dx}\right)^4=0$$

$\therefore$ Order of differential equation = Order of the highest order derivative in the given differential equation = 2

TiPS

- *Degree of the differential equation should be free from radical sign and fractions.*
- *Order and degree (if defined) of a differential equation are always positive integers.*

and degree of differential equation = Power of highest order derivative in the given differential equation = 2.

COMMON ERROR

Some students take degree as the highest power of the derivative.

3. Given differential equation is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=5$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=5$$

Here, order of differential equation = 2

and degree of differential equation = 1

$\therefore$ Required sum = 2 + 1 = 3.

4. (i)

TRICK

If the differential equation is not a polynomial equation in its derivatives, then its degree is not defined.

The degree of differential equation is not defined.

(ii) The degree of differential equation is not defined.

COMMON ERROR

Mostly students write the degree as 1.

5. Order of differential equation = 2

and degree of differential equation = 3

$\therefore$ Required sum = order + degree = 2 + 3 = 5

6. We know that, the number of arbitrary constants in the particular solution of a differential equation is 0.

$\therefore$ Required solution is 0.

7. Given function, $y = ax + 2a^2$ ...(1)

On differentiating w.r.t. 'x', $\frac{dy}{dx}=a$ ...(2)

$$\text{LHS} = 2\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) - y$$

$$= 2(a)^2 + xa - (ax + 2a^2)$$

$$= 2a^2 + ax - ax - 2a^2 = 0 = \text{RHS}$$

Hence, $y = ax + 2a^2$ is a solution of the differential equation $2\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) - y = 0$. **Hence proved.**

8. Given differential equation is

$$\frac{dy}{dx} = 2^{-y} = \frac{1}{2^y}$$

$$\Rightarrow \quad \int 2^y\,dy = \int dx \qquad \text{[on integrating]}$$

$$\Rightarrow \quad \frac{2^y}{\log 2} = x + C_1$$

$$\Rightarrow \quad 2^y = x\log 2 + C_1 \log 2$$

$$\Rightarrow \quad 2^y = x\log 2 + C \qquad \text{where } (C = C_1\log 2)$$

which is the required solution.

COMMON ERROR

Some students evaluate:

$$\int a^x\,dx \neq a^x \log a + C$$

9. Given differential equation is

$$\frac{dy}{dx} = e^{x+y} = e^x \cdot e^y$$

$$\Rightarrow \quad \int e^{-y}\,dy = \int e^x\,dx \qquad \text{[on integrating]}$$

$$\Rightarrow \quad -e^{-y} = e^x + C$$

which is the required general solution.

10. Given differential equation is

$$\frac{dy}{dx} = x^3 \cdot e^{-2y}$$

TRICK

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$$

$$\Rightarrow \quad \int e^{2y}\,dy = \int x^3\,dx \qquad \text{[on integrating]}$$

$$\Rightarrow \quad \frac{e^{2y}}{2} = \frac{x^4}{4} + C_1$$

$$\Rightarrow \quad 2e^{2y} = x^4 + 4C_1$$

$$\Rightarrow \quad 2e^{2y} = x^4 + C \qquad \text{where } (C = 4C_1)$$

which is the required solution.

11. Given, $\frac{dy}{dx} = \frac{1+x^2}{y+\sin y}$

By variable separable method,

$$(y + \sin y)\,dy = (1 + x^2)\,dx$$

$$\Rightarrow \int (y+\sin y)\,dy = \int (1+x^2)\,dx \qquad \text{[on integrating]}$$

$$\Rightarrow \quad \frac{y^2}{2} - \cos y = x + \frac{x^3}{3} + C$$

which is the required solution.

12. Given, $(1-x)\,dy - (3+y)\,dx = 0$

$$\Rightarrow \quad (1-x)\,dy = (3+y)\,dx$$

By variable separable method,

$$\int \frac{dy}{3+y} = \int \frac{dx}{1-x} \qquad \text{[on integrating]}$$

$$\Rightarrow \quad \log|3+y| = -\log|1-x| + \log C$$

$$\Rightarrow \quad \log|(3+y)(1-x)| = \log C$$

$$\Rightarrow \quad (3+y)(1-x) = C$$

which is the required solution.

13. Homogeneous differential equation must have same degree in both numerator and denominator.

$\because$ Degree of denominator = 3

$\therefore$ Degree of numerator = 3

i.e., $n = 3$

Alternate Method:

Given, $\frac{dy}{dx} = \frac{x^3 - y^n}{x^2y + xy^2} = F(x, y)$ (say)

Now, $F(\lambda x, \lambda y) = \frac{(\lambda x)^3 - (\lambda y)^n}{(\lambda x)^2 \lambda y + (\lambda x)((\lambda y)^2}$

$$= \frac{\lambda^3 x^3 - \lambda^n y^n}{\lambda^3 x^2 y + \lambda^3 xy^2}$$

$$= \frac{\lambda^3}{\lambda^3}\left\{\frac{x^3 - \lambda^{n-3} y^n}{x^2 y + xy^2}\right\}$$

$$= \lambda^0 \left\{\frac{x^3 - \lambda^{n-3} . y^n}{x^2 y + xy^2}\right\}$$

If the above differential is homogeneous, then

$$\lambda^{n-3} = 1 = \lambda^0$$

$\Rightarrow$ $n - 3 = 0$

$\Rightarrow$ $n = 3$

14. Given differential equation is $\frac{dy}{dx} + \frac{1}{\sqrt{x}} y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$.

On comparing with $\frac{dy}{dx} + Py = Q$, we get $P = \frac{1}{\sqrt{x}}$

$\therefore$ $\text{IF} = e^{\int P\,dx} = e^{\int \frac{dx}{\sqrt{x}}} = e^{2\sqrt{x}}$

15. Given differential equation is $(\cot y - 2xy)\frac{dy}{dx} = (1 + y^2)$.

$\Rightarrow$ $\frac{dx}{dy} = \frac{\cot y - 2xy}{(1+y^2)}$

$$= \frac{\cot y}{1+y^2} - \frac{2xy}{1+y^2}$$

$\Rightarrow$ $\frac{dx}{dy} + \frac{2y}{1+y^2} \cdot x = \frac{\cot y}{1+y^2}$

On comparing with, $\frac{dx}{dy} + Px = Q$, we get

$$P = \frac{2y}{1+y^2}$$

TR!CK

$$\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + C$$

$\therefore$ $\text{IF} = e^{\int P\,dy} = e^{\int \frac{2y}{1+y^2} dy}$

$$= e^{\log|1+y^2|} = 1 + y^2$$

Short Answer Type-I Questions

1. Given, $ax^2 + by^2 = 1$

On differentiating w.r.t. 'x', we get

$$2ax + 2by\, y_1 = 0$$

$\Rightarrow$ $ax + by\, y_1 = 0$ ...(1)

Again differentiating w.r.t 'x', we get

$$a + b\,\{y\, y_2 + y_1 y_1\} = 0$$

$\Rightarrow$ $a = -b(y\, y_2 + y_1^2)$

Now, put the value of a in eq. (1), we get

$$-b(y\, y_2 + y_1^2)\, x + b\, yy_1 = 0$$

$\Rightarrow$ $-x(y\, y_2 + y_1^2) + yy_1 = 0$ $(\because\ b \neq 0)$

$\therefore$ $x\,(y\, y_2 + y_1^2) = y\, y_1$ **Hence proved.**

COMMON ERR!R

Many students do not have an idea of formation of differential equation and make errors while solving it.

2. Given, $\frac{dy}{dx} + \frac{1+y^2}{x} = 0$

$\Rightarrow$ $\frac{dy}{dx} = -\frac{1+y^2}{x}$

By the variable separable method,

$$\frac{dy}{1+y^2} = -\frac{dx}{x}$$

TR!CK

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$\Rightarrow$ $\int \frac{1}{1+y^2} dy + \int \frac{dx}{x} = 0$ [on integrating]

$\Rightarrow$ $\tan^{-1} y + \log |x| = C$

which is the required solution.

3. Given, $\frac{dy}{dx} = xy + x + y + 1$

$\Rightarrow$ $\frac{dy}{dx} = y\,(x+1) + 1\,(x+1) = (y+1)\,(x+1)$

$\Rightarrow$ $\frac{dy}{(y+1)} = (x+1)\, dx$

$\Rightarrow$ $\int \frac{dy}{(y+1)} = \int (x+1)\, dx$

$\Rightarrow$ $\log |y+1| = \frac{x^2}{2} + x + C$

4. Given differential equation is

$$\log \frac{dy}{dx} = ax + by$$

$\Rightarrow$ $\frac{dy}{dx} = e^{(ax+by)} = e^{ax} \times e^{by}$

$\Rightarrow$ $\frac{dy}{e^{by}} = e^{ax}\, dx$

$\Rightarrow$ $e^{-by}\, dy = e^{ax}\, dx$

$\Rightarrow$ $\int e^{-by}\, dy = \int e^{ax}\, dx$

$\Rightarrow$ $\frac{e^{-by}}{-b} = \frac{e^{ax}}{a} - C$

$\Rightarrow$ $\frac{e^{ax}}{a} + \frac{1}{be^{by}} = C$

5. Given, $\log\left(\frac{dy}{dx}\right) = x - y$

$\Rightarrow$ $\frac{dy}{dx} = e^{x-y} = e^x \cdot e^{-y} = \frac{e^x}{e^y}$

$\Rightarrow$ $\int e^y dy = \int e^x dx$ [on integrating]

$\Rightarrow$ $e^y = e^x + C$

6. Given differential equation is

$$\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}} = \frac{3\,(e^{2x} + e^{2x} \cdot e^{2x})}{e^x + \frac{1}{e^x}}$$

$$= \frac{3e^{2x}\,(1 + e^{2x})}{(e^{2x}+1)} \cdot e^x = 3e^{3x}$$

$\Rightarrow$ $\int dy = 3\int e^{3x}\, dx$ [on integrating]

$\Rightarrow$ $y = \frac{3 \cdot e^{3x}}{3} + C$

$\Rightarrow$ $y = e^{3x} + C$, which is the required solution.

7. Given, $\sec x \operatorname{cosec} y\, dx + \sec y\, dy = 0$

$\Rightarrow \sec x \operatorname{cosec} y\, dx = -\sec y\, dy$

$\Rightarrow \sec x\, dx = -\dfrac{\sec y}{\operatorname{cosec} y} dy = -\dfrac{1/\cos y}{1/\sin y} dy$

$\Rightarrow \int \sec x\, dx = -\int \tan y\, dy$ [on integrating]

$\Rightarrow \log|\sec x + \tan x| = \log|\cos y| + \log C$

$\Rightarrow \log\left|\dfrac{\sec x + \tan x}{\cos y}\right| = \log C$

$\Rightarrow \sec x + \tan x = C \cdot \cos y$

which is the required solution.

8. Given, $(\cos^2 x - \sin^2 x)\, dy + 2 \sin y \cos y\, dx = 0$

$\Rightarrow \cos 2x\, dy + \sin 2y\, dx = 0$

$\Rightarrow \cos 2x\, dy = -\sin 2y\, dx$

$\Rightarrow \dfrac{dy}{\sin 2y} = -\dfrac{dx}{\cos 2x}$

$\Rightarrow \int \operatorname{cosec} 2y\, dy = -\int \sec 2x\, dx$ [on integrating]

TR!CKS

- $\int \sec ax\, dx = \dfrac{1}{a} \log|\sec ax + \tan ax| + C$
- $\int \operatorname{cosec} ax\, dx = \dfrac{1}{a} \log|\operatorname{cosec} ax - \cot ax| + C$

$\Rightarrow \dfrac{\log|\operatorname{cosec} 2y - \cot 2y|}{2} = \dfrac{-\log|\sec 2x + \tan 2x|}{2} + \dfrac{1}{2}\log C$

$\Rightarrow \log|\operatorname{cosec} 2y - \cot 2y| + \log|\sec 2x + \tan 2x| = \log C$

$\Rightarrow (\operatorname{cosec} 2y - \cot 2y)(\sec 2x + \tan 2x) = C$

$\Rightarrow \left(\dfrac{1 - \cos 2y}{\sin 2y}\right)\left(\dfrac{1 + \sin 2x}{\cos 2x}\right) = C$

which is the required solution.

9. Given differential equation is

$\dfrac{dy}{dx} + \dfrac{1+y^2}{y} = 0 \Rightarrow \int \dfrac{y}{1+y^2} dy = -\int dx$

Let $1 + y^2 = t \Rightarrow 2y\, dy = dt$

$\therefore \dfrac{1}{2}\int \dfrac{dt}{t} = -\int dx$

$\Rightarrow \dfrac{1}{2}\log|t| = -x + C_1$

$\Rightarrow \dfrac{1}{2}\log|1+y^2| = -x + C_1$

$\Rightarrow \log|1+y^2| = -2x + C$

$\Rightarrow 2x + \log|1+y^2| = C$ (where $C = 2C_1$)

10. Given differential equation is

$\dfrac{e^x}{1+e^x} dx = \dfrac{y}{y+1} dy$

$\Rightarrow \int \dfrac{e^x}{1+e^x} dx = \int \dfrac{(y+1)-1}{y+1} dy$ [on integrating]

$\Rightarrow \int \dfrac{e^x}{1+e^x} dx = \int \left[1 - \dfrac{1}{y+1}\right] dy$

TR!CK

$\int \dfrac{f'(x)}{f(x)} dx = \log|f(x)| + C$

$\Rightarrow \log|1+e^x| = y - \log|y+1| + C$

11. Given, $(xy^2 + x)\, dx + (yx^2 + y)\, dy = 0$

$\Rightarrow x(y^2+1)\, dx + y(x^2+1)\, dy = 0$

$\Rightarrow \int \dfrac{x}{1+x^2} dx + \int \dfrac{y}{1+y^2} dy = 0$ [on integrating]

TR!CK

$\int \dfrac{f'(x)}{f(x)} dx = \log|f(x)| + C$

$\Rightarrow \dfrac{1}{2}\log|1+x^2| + \dfrac{1}{2}\log|1+y^2| = \dfrac{1}{2}\log C$

$\Rightarrow \dfrac{1}{2}\log|(1+x^2)(1+y^2)| = \dfrac{1}{2}\log C$

$\Rightarrow (1+x^2)(1+y^2) = C$

12. Given, $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

$\Rightarrow \sec^2 x \cdot \tan y\, dx = -\sec^2 y \tan x\, dy$

$\Rightarrow \dfrac{\sec^2 x}{\tan x} dx = -\dfrac{\sec^2 y}{\tan y} dy$

$\Rightarrow \int \dfrac{\sec^2 x}{\tan x} dx + \int \dfrac{\sec^2 y}{\tan y} dy = 0$ [on integrating]

$\Rightarrow \log|\tan x| + \log|\tan y| = \log C$

TR!CK

$\int \dfrac{f'(x)}{f(x)} dx = \log|f(x)|$

$\Rightarrow \log|\tan x \cdot \tan y| = \log C \Rightarrow \tan x \cdot \tan y = C$

which is the required solution.

13. Given, $\dfrac{d^2y}{dx^2} = \sin x$

Integrating both sides w.r.t. 'x', we get

$\dfrac{dy}{dx} = \int \sin x\, dx = -\cos x + C$

Again integrating w.r.t. 'x', we get

$y = -\int \cos x\, dx + C\int 1\, dx$

$\Rightarrow y = -\sin x + Cx + d$

which is the required solution.

14. Given, $\dfrac{dy}{dx} = x^3 \operatorname{cosec} y$.

By variable separable method,

$\dfrac{dy}{\operatorname{cosec} y} = x^3\, dx$

$\Rightarrow \int \sin y\, dy = \int x^3\, dx$ [on integrating]

$\Rightarrow -\cos y = \dfrac{x^4}{4} + C$ ...(1)

Put $y(0) = 0$, then

$-\cos 0 = \dfrac{0}{4} + C$

$\Rightarrow -1 = 0 + C \Rightarrow C = -1$

$\therefore -\cos y = \dfrac{x^4}{4} - 1$ [from eq. (1)]

$\Rightarrow \cos y = 1 - \dfrac{x^4}{4}$

which is the required solution.

COMMON ERR!R

Students forget to find the particular solution after finding the general solution.

15. Given, $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

TiP

Practice more problems based on finding particular solution.

$$\Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} \quad \text{[on integrating]}$$

$$\Rightarrow \tan^{-1} y = \tan^{-1} x + C \quad ...(1)$$

TR!CK

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Now putting $x = 0$ and $y = \sqrt{3}$, we get

$$\tan^{-1}\sqrt{3} = \tan^{-1} 0 + C$$

$$\Rightarrow \frac{\pi}{3} = 0 + C \Rightarrow C = \frac{\pi}{3}$$

Put the value of C in eq. (1), we get

$$\tan^{-1} y = \tan^{-1} x + \frac{\pi}{3}$$

$$\Rightarrow y = \tan\left(\tan^{-1} x + \frac{\pi}{3}\right)$$

$$\Rightarrow y = \frac{\tan(\tan^{-1} x) + \tan\frac{\pi}{3}}{1 - \tan(\tan^{-1} x)\tan\frac{\pi}{3}}$$

$$\Rightarrow y = \frac{x+\sqrt{3}}{1-x\sqrt{3}} \Rightarrow y - xy\sqrt{3} = x + \sqrt{3}$$

$$\Rightarrow y - x = \sqrt{3}(1+xy)$$

which is the required solution.

16. Given, $\frac{dy}{dx} + \frac{2}{x}y = x$

On comparing with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{2}{x} \text{ and } Q = x$$

TR!CK

$m \log n = \log n^m$, $e^{\log a} = a$

Now,

$$\text{IF} = e^{\int P\,dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$$

$\therefore$ Complete solution is

$$y\,(\text{IF}) = \int Q\,(\text{IF})\,dx + C$$

$$\Rightarrow y \cdot x^2 = \int x \cdot x^2\,dx + C = \int x^3\,dx + C$$

$$\Rightarrow y \cdot x^2 = \frac{x^4}{4} + C \Rightarrow y = \frac{x^2}{4} + \frac{C}{x^2}$$

17. Given, $\frac{dy}{dx} + 2y = e^{3x}$

On comparing with $\frac{dy}{dx} + Py = Q$, we get

$$P = 2 \text{ and } Q = e^{3x}$$

Now,

$$\text{IF} = e^{\int P\,dx} = e^{\int 2\,dx} = e^{2x}$$

$\therefore$ Complete solution is $y \cdot (\text{IF}) = \int Q \cdot (\text{IF})\,dx + C$

$$y \cdot e^{2x} = \int e^{3x} \cdot e^{2x}\,dx + C$$

$$\Rightarrow ye^{2x} = \int e^{5x}\,dx + C$$

$$\Rightarrow ye^{2x} = \frac{1}{5}e^{5x} + C$$

$$\Rightarrow y = \frac{e^{3x}}{5} + Ce^{-2x}$$

COMMON ERR(!)R

Many students forget to express the answer in terms of constant 'C'.

18. Given differential equation is

$$x\frac{dy}{dx} - y = x^2 \cdot e^x$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{x} \cdot y = x \cdot e^x$$

On comparing with $\frac{dy}{dx} + Py = Q$, we get

$$P = -\frac{1}{x} \text{ and } Q = x \cdot e^x$$

$$\therefore \text{IF} = e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

So, required solution is

$$y \cdot (\text{IF}) = \int Q \cdot (\text{IF})\,dx + C$$

$$\Rightarrow y \cdot \frac{1}{x} = \int x \cdot e^x \cdot \frac{1}{x}\,dx + C$$

$$\Rightarrow \frac{y}{x} = \int e^x dx + C = e^x + C$$

$$\Rightarrow y = xe^x + Cx \quad ...(1)$$

When $x = 1$, then $y = 0$.

$$\therefore 0 = 1 \cdot e^1 + C \Rightarrow C = -e$$

Put the value 'C' in eq. (1), we get

$$y = xe^x - ex$$

Short Answer Type-II Questions

1. Given differential equation is

$$\frac{dy}{dx} = \cos(x+y) + \sin(x+y) \quad ...(1)$$

Let $x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$

$$\therefore \frac{dt}{dx} - 1 = \cos t + \sin t \quad \left[\text{put the value of } \frac{dy}{dx} \text{ in eq. (1)}\right]$$

$$\Rightarrow \frac{dt}{dx} = 1 + \cos t + \sin t$$

$$\Rightarrow \frac{dt}{dx} = 2\cos^2\frac{t}{2} + 2\sin\frac{t}{2}\cot\frac{t}{2}$$

$$\Rightarrow \int \frac{dt}{2\left(\cos^2\frac{t}{2} + \sin\frac{t}{2}\cot\frac{t}{2}\right)} = \int dx$$

TR!CK

$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$, $\sin\theta = 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}$

$$\Rightarrow \frac{1}{2}\int \frac{\sec^2\frac{t}{2}\,dt}{\left(1+\tan\frac{t}{2}\right)} = x + C$$

Let $\quad \tan\frac{t}{2} = u \Rightarrow \frac{1}{2}\sec^2\frac{t}{2}dt = du$

$\therefore \quad \int \frac{du}{1+u} = x + C \quad \Rightarrow \log|(1+u)| = x + C$

$\Rightarrow \quad \log\left|1+\tan\frac{t}{2}\right| = x + C$

$\Rightarrow \quad \log\left|1+\tan\left\{\frac{x+y}{2}\right\}\right| = x + C$

2. Given differential equation is

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1$$

$\Rightarrow \quad (x+1)\frac{dy}{dx} = \frac{2}{e^y} - 1$

$\Rightarrow \quad (x+1)\frac{dy}{dx} = \frac{2-e^y}{e^y}$

$\Rightarrow \quad \frac{e^y}{2-e^y}dy = \frac{dx}{x+1}$

$\Rightarrow \quad -\int \frac{e^y}{e^y-2}dy = \int \frac{dx}{x+1}$ [on integrating]

TR!CK

$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$

$\Rightarrow \quad -\log|e^y - 2| = \log|x+1| + C$ ...(1)

On putting $x = 0$ and $y = 0$, we get

$-\log|e^0 - 2| = \log|0+1| + C$

$\Rightarrow \quad -\log|1-2| = \log(1) + C$

$\Rightarrow \quad -\log|-1| = 0 + C$

$\Rightarrow \quad -\log(1) = 0 + C$

$\Rightarrow \quad -0 = 0 + C \Rightarrow C = 0$

Put the value of C in eq. (1), we get

$-\log|e^y - 2| = \log|x+1| + 0$

$\Rightarrow \quad \log\left|\frac{1}{e^y - 2}\right| = \log|x+1|$

$\Rightarrow \quad \frac{1}{e^y - 2} = (x+1)$

$\Rightarrow \quad e^y = \frac{1}{x+1} + 2$

$\Rightarrow \quad y = \log\left|2 + \frac{1}{x+1}\right|$

which is the required solution.

3. Given differential equation is

$e^x \tan y\, dx + (1-e^x)\sec^2 y\, dy = 0$

$\Rightarrow \quad e^x \tan y\, dx = (e^x - 1)\sec^2 y\, dy$

$\Rightarrow \quad \frac{e^x}{e^x - 1}dx = \frac{\sec^2 y}{\tan y}dy$

$\Rightarrow \quad \int \frac{e^x}{e^x-1}dx = \int \frac{\sec^2 y}{\tan y}dy$ [on integrating]

$\Rightarrow \quad \log|e^x - 1| = \log|\tan y| + C$ ...(1)

$\left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C\right]$

4. We know that, slope of the tangent of any curve is equal to $\frac{dy}{dx}$.

$\therefore \quad \frac{dy}{dx} = \frac{2x}{y^2}$

TIP

Practice more problems based on finding particular solution.

$\Rightarrow \quad y^2\, dy = 2x\, dx$

$\Rightarrow \quad \int y^2\, dy = 2\int x\, dx$ [on integrating]

$\Rightarrow \quad \frac{y^3}{3} = 2 \times \frac{x^2}{2} + C$

$\Rightarrow \quad \frac{y^3}{3} = x^2 + C$ ...(1)

Put $x = -2$ and $y = 3$ in eq. (1), we get

$\frac{(3)^3}{3} = (-2)^2 + C$

$\Rightarrow \quad \frac{27}{3} = 4 + C \Rightarrow C = 9 - 4 \quad \Rightarrow \quad C = 5$

Put the value of C in eq. (1), we get

$\frac{y^3}{3} = x^2 + 5 \Rightarrow y = (3x^2 + 15)^{1/3}$

which is the required equation of curve.

COMMON ERR(!)R

Students forget to find the particular solution after finding the general solution.

5. Given differential equation is

$x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$

$\Rightarrow \quad x\,dy = (y + \sqrt{x^2 + y^2})\,dx$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x}$ ...(1)

which is a homogeneous differential equation.

Now put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq. (1), we get

$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{vx + x\sqrt{1+v^2}}{x} - v$

$\Rightarrow \quad x\frac{dv}{dx} = v + \sqrt{1+v^2} - v = \sqrt{1+v^2}$

TR!CK

$\int \frac{dx}{\sqrt{a^2 + x^2}} = \log|x + \sqrt{a^2 + x^2}| + C$

$\Rightarrow \quad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

$\Rightarrow \quad \int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$ [on integrating]

$\Rightarrow \quad \log|v + \sqrt{1+v^2}| = \log|x| + \log k$ $[\because k > 0]$

$\Rightarrow \quad \log\left|\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right| = \log|x| + \log k$ $\left[\because v = \frac{y}{x}\right]$

$\Rightarrow \quad \log|y+\sqrt{x^2+y^2}|-\log|x| = \log|x|+\log k$

$\Rightarrow \quad \log|y+\sqrt{x^2+y^2}| = 2\log|x|+\log k$

$\Rightarrow \quad \log|y+\sqrt{x^2+y^2}| = \log kx^2$

$\Rightarrow \quad y+\sqrt{x^2+y^2} = kx^2$

$\therefore \quad y+\sqrt{x^2+y^2} = Cx^2$, where $C = k$

which is the required general solution.

6. Given differential equation is

$$x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{y}{x}\right) - \sin\left(\frac{y}{x}\right) \quad ...(1)$$

which is a homogeneous differential equation.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq. (1), we get

$$v + x\frac{dv}{dx} = v - \sin v$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{-\sin v}{x}$$

$$\Rightarrow \quad \frac{dv}{\sin v} = \frac{-dx}{x}$$

$$\Rightarrow \quad \int \operatorname{cosec} v\, dv = -\int \frac{dx}{x} \quad \text{[on integrating]}$$

$$\Rightarrow \quad \log\left|\tan\frac{v}{2}\right| = -\log|x| + \log C$$

where $\log C$ is an arbitrary constant.

$$\Rightarrow \quad \log\left|x\tan\frac{v}{2}\right| = \log C \quad \left[\because v = \frac{y}{x}\right]$$

$$\Rightarrow \quad x\tan\frac{y}{2x} = C$$

which is the required general solution.

7. Given differential equation is

$$x\frac{dy}{dx} = y(\log y - \log x + 1)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\left\{\log\left(\frac{y}{x}\right)+1\right\} \quad ...(1)$$

which is a homogeneous differential equation.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq. (1), we get

$$v + x\frac{dv}{dx} = \frac{vx}{x}\left\{\log\left(\frac{vx}{x}\right)+1\right\}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v(\log v + 1)$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v\log v + v \Rightarrow x\frac{dv}{dx} = v\log v$$

$$\Rightarrow \quad \int \frac{dv}{v\log v} = \int \frac{dx}{x} \quad \text{[on integrating]}$$

$$\Rightarrow \quad \int \frac{(1/v)}{\log v}dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{d(\log v)}{\log v} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \log|\log v| = \log|x| + \log C$$

$$\Rightarrow \quad \log\left|\log\left(\frac{y}{x}\right)\right| = \log Cx \quad \Rightarrow \quad \log\left(\frac{y}{x}\right) = Cx$$

$$\therefore \quad y = xe^{Cx}$$

which is the required solution.

8. Given differential equation is $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$. ...(1)

which is a homogeneous differential equation.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq. (1), we get

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2x(vx)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{x^2(1+v^2)}{2x^2v} - v = \frac{1+v^2}{2v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$$

$$\Rightarrow \quad \frac{2v}{1-v^2}dv = \frac{dx}{x}$$

$$\Rightarrow \quad -\int \frac{-2v}{1-v^2}dv = \int \frac{dx}{x} \quad \text{[on integrating]}$$

TR!CK

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$$

$$\Rightarrow \quad -\log|1-v^2| = \log|x| - \log C$$

$$\Rightarrow \quad -\log\left|1-\frac{y^2}{x^2}\right| = \log|x| - \log C \quad [\because y = vx]$$

$$\Rightarrow -\log\left|\frac{x^2-y^2}{x^2}\right| = \log|x| - \log C$$

$$\Rightarrow \quad -\log|x^2-y^2| + \log x^2 = \log|x| - \log C$$

$$\Rightarrow \quad -\log|x^2-y^2| + 2\log|x| = \log|x| - \log C$$

$$\Rightarrow \quad \log|x| - \log|x^2-y^2| = -\log C$$

$$\Rightarrow \quad \log\left|\frac{x}{x^2-y^2}\right| = \log\left|\frac{1}{C}\right|$$

$$\Rightarrow \quad \frac{x}{x^2-y^2} = \frac{1}{C}$$

$\therefore \quad (x^2-y^2) = Cx$ **Hence proved.**

9. Given, $(3xy+y^2)\,dx + (x^2+xy)\,dy = 0$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{3xy+y^2}{x^2+xy}\right) \quad ...(1)$$

which is a homogeneous differential equation.

TiP

In homogeneous differential equation, the degree of all terms will be the same.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq. (1), we get

$$-\left(\frac{3vx^2+v^2x^2}{x^2+x^2v}\right) = v + x\frac{dv}{dx}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\left[\frac{3v+v^2}{1+v}+v\right] = -\frac{2v^2+4v}{1+v}$$

$$\Rightarrow \quad \frac{1+v}{2(v^2+2v)} = -\frac{dx}{x}$$

$$\Rightarrow \quad \frac{2v+2}{2(v^2+2v)}dv = -2\frac{dx}{x}$$

$\Rightarrow \quad \frac{1}{2}\int \frac{2v+2}{v^2+2v}dv = -\int \frac{2}{x}dx$ [on integrating]

TR!CK

$\int \frac{f'(x)}{f(x)}dx = \log|f(x)|+C$

$\Rightarrow \frac{1}{2}\log|v^2+2v| = -2\log x + \log\sqrt{C} = \log\left|\frac{\sqrt{C}}{x^2}\right|$

$\Rightarrow \quad \sqrt{v^2+2v} = \frac{\sqrt{C}}{x^2}$

$\Rightarrow \quad \frac{y^2}{x^2}+\frac{2y}{x} = \frac{C}{x^4}$ [put $y = vx$ and squaring]

$\Rightarrow \quad x^2y^2+2x^3y = C$ or $x^2y(y+2x) = C$

10. Given differential equation is

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right)+x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right)+x}{x\cos\left(\frac{y}{x}\right)} \quad ...(1)$$

which is a homogeneous differential equation.

TiP

A differential equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is said to be homogeneous, if $f(x, y)$ and $g(x, y)$ are homogeneous functions of same degree.

Now putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq. (1), we get

$$v + x\frac{dv}{dx} = \frac{(vx)\cos\left(\frac{vx}{x}\right)+x}{x\cos\left(\frac{vx}{x}\right)}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{x\{v\cos v+1\}}{x\cos v} = \frac{v\cos v+1}{\cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v+1}{\cos v} - v = \frac{v\cos v+1-v\cos v}{\cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1}{\cos v}$$

$$\Rightarrow \quad \int\cos v\, dv = \int\frac{dx}{x} \quad \text{[on integrating]}$$

$$\Rightarrow \quad \sin v = \log|x|+C$$

$$\Rightarrow \quad \sin\frac{y}{x} = \log|x|+C$$

which is the required general solution.

11. Given differential equation is

$$ye^{x/y}\,dx = (xe^{x/y}+y^2)\,dy,\ y \neq 0$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{(xe^{x/y}+y^2)}{ye^{x/y}} \quad ...(1)$$

which is a homogeneous differential equation.

Now putting $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ in eq. (1), we get

$$v + y\frac{dv}{dy} = \frac{vye^{vy/y}+y^2}{ye^{vy/y}}$$

$$\Rightarrow \quad v + y\frac{dv}{dy} = \frac{y(ve^v+y)}{ye^v}$$

$$\Rightarrow \quad v + y\frac{dv}{dy} = \frac{ve^v+y}{e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{ve^v+y}{e^v} - v = \frac{ve^v+y-ve^v}{e^v}$$

$$\Rightarrow \quad y\cdot\frac{dv}{dy} = \frac{y}{e^v} \Rightarrow e^v\,dv = \frac{y}{y}dy$$

$$\Rightarrow \quad \int e^v dv = \int dy \quad \text{[on integrating]}$$

$$\Rightarrow \quad e^v = y + C$$

$$\Rightarrow \quad e^{x/y} = y + C \quad [\because x = vy]$$

which is the required solution.

12. Given differential equation is

$$2y\,e^{x/y}dx + (y - 2x\,e^{x/y})\,dy = 0$$

$$\Rightarrow \quad (2x\,e^{x/y} - y)\,dy = 2y\,e^{x/y}\,dx$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{2x\,e^{x/y}-y}{2y\,e^{x/y}} \quad ...(1)$$

which is a homogeneous differential equation.

Put $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ in eq. (1), we get

$$v + y\cdot\frac{dv}{dy} = \frac{2(vy)e^{vy/y}-y}{2y\,e^{vy/y}} = \frac{y(2v\,e^v-1)}{2y\,e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{2ve^v-1}{2e^v} - v = \frac{2ve^v-1-2ve^v}{2e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = -\frac{1}{2e^v}$$

$$\Rightarrow \quad 2e^v\,dv = -\frac{dy}{y}$$

$$\Rightarrow \quad 2\int e^v\,dv = -\int\frac{dy}{y} \quad \text{[on integrating]}$$

$$\Rightarrow \quad 2e^v = -\log|y|+C$$

$$\Rightarrow \quad 2e^{x/y} = -\log|y|+C \quad ...(2)$$

Now putting $y = 1$ and $x = 0$ in eq. (2), we get

$$2e^0 = -\log(1) + C$$

$$\Rightarrow \quad 2\times1 = -0 + C \Rightarrow C = 2$$

From eq. (2), we get

$$2e^{x/y} = -\log|y|+2$$

$$\Rightarrow \quad 2e^{x/y} + \log|y| = 2$$

COMMON ERROR

Students forget to find the particular solution after finding the general solution.

13. Given differential equation is

$$x\sin\left(\frac{y}{x}\right)\frac{dy}{dx} + x - y\sin\left(\frac{y}{x}\right) = 0$$

TiP

A differential equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is said to be homogeneous, if f(x, y) and g (x, y) are homogeneous functions of same degree.

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) - x}{x\cdot\sin\left(\frac{y}{x}\right)} \quad \ldots(1)$$

which is a homogeneous differential equation.

Now, put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq. (1), we get

$$v + x\frac{dv}{dx} = \frac{vx\cdot\sin\left(\frac{vx}{x}\right) - x}{x\cdot\sin\left(\frac{vx}{x}\right)}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{x\{v\sin v - 1\}}{x\cdot\sin v} = \frac{v\sin v - 1}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\sin v - 1 - v\sin v}{\sin v} = \frac{-1}{\sin v}$$

$$\Rightarrow \quad \int \sin v\, dv = -\int \frac{dx}{x} \quad \text{[on integrating]}$$

$$\Rightarrow \quad -\cos v = -\log|x| + C$$

$$\Rightarrow \quad -\cos\left(\frac{y}{x}\right) = -\log|x| + C \quad [\because y = vx] \ldots(2)$$

Putting $x = 1$ and $y = \frac{\pi}{2}$ we get

$$-\cos\left(\frac{\pi/2}{1}\right) = -\log(1) + C$$

$$\Rightarrow \quad -\cos\frac{\pi}{2} = -0 + C$$

$$\Rightarrow \quad -0 = -0 + C \Rightarrow C = 0.$$

Put the value of C in eq. (2), we get

$$-\cos\left(\frac{y}{x}\right) = -\log|x| + 0$$

$$\therefore \quad \cos\left(\frac{y}{x}\right) = \log|x|$$

which is the required solution.

14. Given, $(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{x^2-1}\cdot y = \frac{2}{(x^2-1)^2}$$

which is of the form $\frac{dy}{dx} + Py = Q$,

where $\quad P = \frac{2x}{x^2-1}$ and $Q = \frac{2}{(x^2-1)^2}$

TR!CK

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$$

$$\therefore \quad \text{IF} = e^{\int P\,dx} = e^{\int \frac{2x}{x^2-1}dx} = e^{\log|x^2-1|} = x^2-1$$

Therefore, the complete solution is

$$y\cdot(\text{IF}) = \int Q\cdot(\text{IF})\,dx + C$$

$$\Rightarrow \quad y\times(x^2-1) = \int\left((x^2-1)\times\frac{2}{(x^2-1)^2}\right)dx + C$$

$$\Rightarrow \quad y(x^2-1) = \int\frac{2}{x^2-1}dx = 2\cdot\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

$$\Rightarrow \quad y = \frac{1}{x^2-1}\log\left|\frac{x-1}{x+1}\right| + \frac{C}{x^2-1}$$

which is the required solution.

15. Given differential equation is

$$\frac{dy}{dx} - \frac{2x}{1+x^2}y = x^2+2$$

On comparing with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{-2x}{1+x^2} \text{ and } Q = x^2+2$$

TR!CK

$$\int\frac{f'(x)}{f(x)}dx = \log|f(x)|$$

$$\therefore \quad \text{IF} = e^{\int P\,dx} = e^{\int\frac{-2x}{1+x^2}dx} = e^{-\log|1+x^2|}$$

$$= e^{\log\left|\frac{1}{1+x^2}\right|} = \frac{1}{1+x^2}$$

So, solution of linear differential equation is

$$y\cdot(\text{IF}) = \int Q\cdot(\text{IF})\,dx + C$$

$$\Rightarrow \quad y\cdot\frac{1}{1+x^2} = \int(x^2+2)\cdot\frac{1}{(x^2+1)}dx + C$$

$$\Rightarrow \quad \frac{y}{1+x^2} = \int\frac{(x^2+1)+1}{(x^2+1)}dx + C$$

TR!CK

$$\int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\Rightarrow \quad \frac{y}{1+x^2} = \int\left\{1 + \frac{1}{1+x^2}\right\}dx + C$$

$$\Rightarrow \quad \frac{y}{1+x^2} = \int dx + \int\frac{dx}{1+x^2}dx + C$$

$$\Rightarrow \quad \frac{y}{1+x^2} = x + \tan^{-1}x + C$$

$$\Rightarrow \quad y = (1+x^2)(x+\tan^{-1}x) + C(1+x^2)$$

16. Given differential equation is

$$\frac{dy}{dx} = \frac{x+y}{x},\ y(1) = 0$$

or $\quad \frac{dy}{dx} - \frac{y}{x} = 1$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Here $P = -\frac{1}{x}$ and $Q = 1$

$$\therefore \quad \text{IF} = e^{\int P dx} = e^{\int -\frac{1}{x}dx} = e^{-\log|x|}$$

$$= e^{\log\left|\frac{1}{x}\right|} = \frac{1}{x}$$

$\therefore$ Complete solution is

$$y\times\text{IF} = \int Q\cdot(\text{IF})\,dx + C$$

$$y\times\frac{1}{x} = \int\frac{1}{x}\times 1\,dx + C$$

$$\Rightarrow \quad \frac{y}{x} = \log|x| + C$$

$$\Rightarrow \quad y = x(\log|x| + C)$$

$$\because \quad y(1) = 0$$

Put $x = 1$ and $y = 0$, we get

$$0 = 1(\log 1 + C) \Rightarrow C = 0$$

$\therefore$ Required particular solution is $y = x\log|x|$.

17. Given, $(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\tan^{-1} y - x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} - \frac{x}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{1+y^2}\cdot x = \frac{\tan^{-1} y}{1+y^2} \qquad \ldots(1)$$

On comparing with $\frac{dx}{dy} + Px = Q$, we get

$$P = \frac{1}{1+y^2} \quad \text{and} \quad Q = \frac{\tan^{-1} y}{1+y^2}$$

$$\therefore \quad \text{IF} = e^{\int P\, dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

So, complete solution is

$$x \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dy + C$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \times e^{\tan^{-1} y} dy + C$$

Let $t = \tan^{-1} y \Rightarrow dt = \frac{1}{1+y^2} dy$

$$\therefore \quad x \cdot e^{\tan^{-1} y} = \int t \cdot e^t dt + C$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = t \int e^t dx - \int \left\{ \frac{d}{dt}(t) \int e^t dx \right\} dt + C$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = t \cdot e^t - \int (1 \cdot e^t)\, dt + C = t \cdot e^t - e^t + C$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = \tan^{-1} y \cdot e^{\tan^{-1} y} - e^{\tan^{-1} y} + C$$

$$\therefore \quad x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$$

which is the required solution.

COMMON ERROR

Some students could not recognize the form of differential equation correctly.

18. Given differential equation is

$$y\, dx + (x - y^2)\, dy = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{(y^2 - x)}{y} = y - \frac{x}{y}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{y} \cdot x = y$$

which is of the form $\frac{dx}{dy} + Px = Q$.

Here, $P = \frac{1}{y}$ and $Q = y$

$$\therefore \quad \text{IF} = e^{\int P dy} = e^{\int \frac{1}{y} dy} = e^{\log |y|} = y$$

So, complete solution is

$$x \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dy + C$$

$$\Rightarrow \quad x \cdot y = \int y \cdot y\, dy + C = \int y^2 dy + C$$

$$\Rightarrow \quad xy = \frac{y^3}{3} + C$$

which is the required general solution.

19. Given differential equation is $\frac{dy}{dx} - y = \sin x$.

On comparing with $\frac{dy}{dx} + Py = Q$, we get

$$P = -1 \quad \text{and} \quad Q = \sin x$$

$$\therefore \quad \text{IF} = e^{\int P\, dx} = e^{\int -1 dx} = e^{-x}$$

So, complete solution is

$$y \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dx + C$$

$$\Rightarrow \quad y \cdot e^{-x} = \int \sin x \cdot e^{-x}\, dx + C \qquad \ldots(1)$$

TRICK

Integration by parts,

$$\int \underset{\text{I}}{u}\, \underset{\text{II}}{v}\, dx = u \int v dx - \int \left\{ \frac{d}{dx} u \int v dx \right\} dx,$$

where u and v are the functions of x.

Let $\quad I = \int e^{-x} \cdot \sin x\, dx$

$$\Rightarrow \quad I = \sin x \int e^{-x}\, dx - \int \left\{ \frac{d}{dx} \sin x \int e^{-x}\, dx \right\} dx$$

$$\Rightarrow \quad I = \sin x\, (-e^{-x}) - \int \cos x\, (-e^{-x})\, dx$$

$$\Rightarrow \quad I = -e^{-x} \cdot \sin x + \int e^{-x} \cdot \cos x\, dx$$

$$\Rightarrow \quad I = -e^{-x} \cdot \sin x + \left\{ \cos x \int e^{-x} dx - \int \left(\frac{d}{dx} \cos x \int e^{-x} dx \right) dx \right\}$$

$$\Rightarrow \quad I = -e^{-x} \cdot \sin x + \{ \cos x \cdot (-e^{-x}) - \int (-\sin x)(-e^{-x})\, dx \}$$

$$\Rightarrow \quad I = -e^{-x} \cdot \sin x - e^{-x} \cdot \cos x - \int e^{-x} \cdot \sin x\, dx$$

$$\Rightarrow \quad I = -e^{-x} \sin x - e^{-x} \cdot \cos x - I$$

$$\Rightarrow \quad 2I = -e^{-x} (\cos x + \sin x)$$

$$\therefore \quad I = \int e^{-x} \cdot \sin x\, dx = \frac{-e^{-x}}{2} (\cos x + \sin x)$$

From eq. (1), we get

$$y \cdot e^{-x} = \frac{-e^{-x}}{2} (\cos x + \sin x) + C$$

$$\Rightarrow \quad y = \frac{-1}{2} (\cos x + \sin x) + Ce^{x}$$

20. Given differential equation is

$$(1 + x^2) \frac{dy}{dx} + 2xy - 4x^2 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{1+x^2} \cdot y = \frac{4x^2}{1+x^2}$$

On comparing with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{2x}{1+x^2} \quad \text{and} \quad Q = \frac{4x^2}{1+x^2}$$

TRICK

$$\int \frac{f'(x)}{f(x)} dx = \log | f(x) | + C$$

$$\therefore \quad \text{IF} = e^{\int P\, dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log |1+x^2|} = 1 + x^2$$

So, general solution is

$$y \cdot (\text{IF}) = \int Q \cdot (\text{IF})\, dx + C$$

$$\Rightarrow \quad y \cdot (1 + x^2) = \int \frac{4x^2}{(1+x^2)} \times (1 + x^2)\, dx + C$$

$$\Rightarrow \quad y \cdot (1 + x^2) = 4 \int x^2\, dx + C$$

$$\Rightarrow \quad y \cdot (1 + x^2) = 4 \cdot \frac{x^3}{3} + C \qquad \ldots(1)$$

Now, putting $x = 0$ and $y = 0$ in eq. (1), we get

$$0 \cdot (1 + 0) = 4 \cdot \frac{0}{3} + C \Rightarrow C = 0$$

Put the value of C in eq. (1), we get

$$y\cdot(1+x^2)=4\cdot\frac{x^3}{3}+0$$

$$\therefore \quad 3y\,(1+x^2)=4x^3 \text{ or } y=\frac{4x^3}{3\,(1+x^2)}$$

which is the required solution.

21. Given differential equation is

$$\frac{dy}{dx}+2y\tan x=\sin x$$

On comparing with $\frac{dy}{dx}+Py=0$, we get

$P=2\tan x$ and $Q=\sin x$

$$\therefore \quad \text{IF}=e^{\int P\,dx}=e^{\int 2\tan x\,dx}=e^{2\log|\sec x|}$$
$$=e^{\log\sec^2 x}=\sec^2 x$$

Now, complete solution is

$$y\cdot(\text{IF})=\int Q\cdot(\text{IF})\,dx+C$$
$$\Rightarrow \quad y\cdot\sec^2 x=\int \sin x\cdot\sec^2 x\,dx+C$$
$$\Rightarrow \quad y\cdot\sec^2 x=\int \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}\,dx+C$$
$$\Rightarrow \quad y\cdot\sec^2 x=\int \sec x\cdot\tan x\,dx+C$$

TR!CK

$\int \sec x\cdot\tan x\,dx=\sec x+C$

$$\Rightarrow \quad y.\sec^2 x=\sec x+C \qquad \text{...(1)}$$

Putting $y=0$ and $x=\frac{\pi}{3}$, we get

$$0\cdot\sec^2\left(\frac{\pi}{3}\right)=\sec\frac{\pi}{3}+C$$
$$\Rightarrow \quad 0=2+C$$
$$\Rightarrow \quad C=-2$$

From eq. (1), we get

$$y\sec^2 x=\sec x-2 \quad\Rightarrow\quad y=\frac{\sec x-2}{\sec^2 x}$$
$$\therefore \quad y=\cos x-2\cos^2 x$$

which is the required particular solution.

22. Given differential equation is

$$\frac{dy}{dx}+y\cot x=2x+x^2\cot x$$

On comparing with $\frac{dy}{dx}+Py=Q$, we get

$$P=\cot x \text{ and } Q=2x+x^2\cot x$$
$$\therefore \quad \text{IF}=e^{\int P\,dx}=e^{\int\cot x\,dx}$$
$$=e^{\log|\sin x|}=\sin x$$

So, complete solution is

$$y\,(\text{IF})=\int Q\cdot(\text{IF})\,dx+C$$
$$\Rightarrow y\cdot\sin x=\int(2x+x^2\cot x)\sin x\,dx+C$$
$$=\int(2x\sin x+x^2\cos x)\,dx+C$$
$$=\int 2x\sin x\,dx+\int \underset{\text{I}}{x^2}\underset{\text{II}}{\cos x}\,dx+C$$

TR!CK

Integration by parts,

$$\int uv\,dx=u\int v\,dx-\int\left\{\frac{d}{dx}u\int v dx\right\}dx,$$

where u and v are the functions of x.

$$=\int 2x\sin x\,dx+(x^2)\int\cos x\,dx$$
$$-\int\left\{\frac{d}{dx}x^2\int\cos x\,dx\right\}dx+C$$
$$=\int 2x\sin x\,dx+x^2\sin x-\int 2x\sin x\,dx+C$$
$$\therefore \quad y\sin x=x^2\sin x+C \qquad \text{...(1)}$$

Now put $x=\frac{\pi}{2}$ and $y=0$ in eq. (1), we get

$$0=\left(\frac{\pi}{2}\right)^2\cdot\sin\frac{\pi}{2}+C$$
$$\Rightarrow \quad C=\frac{-\pi^2}{4}\times 1=\frac{-\pi^2}{4}$$

Put the value of C in eq. (1), we get

$$y\sin x=x^2\sin x-\frac{\pi^2}{4} \quad\text{or}\quad y=x^2-\frac{\pi^2}{4\sin x}$$

which is the required particular solution.

23. Given differential equation is

$$x\frac{dy}{dx}+x\cos^2\left(\frac{y}{x}\right)=y$$
$$\Rightarrow \quad \frac{dy}{dx}=\frac{y}{x}-\cos^2\left(\frac{y}{x}\right) \qquad \text{...(1)}$$

which is a homogeneous differential equation.

Put $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in eq. (1), we get

$$v+x\frac{dv}{dx}=\frac{vx}{x}-\cos^2\left(\frac{vx}{x}\right)$$
$$\Rightarrow \quad v+x\frac{dv}{dx}=v-\cos^2 v\Rightarrow x\frac{dv}{dx}=-\cos^2 v$$
$$\Rightarrow \quad \frac{dv}{\cos^2 v}=-\frac{dx}{x}$$
$$\Rightarrow \quad \int\sec^2 v\,dv=-\int\frac{dx}{x}$$
$$\Rightarrow \quad \tan v=-\log|x|+C \qquad \text{[on integrating]}$$
$$\Rightarrow \quad \tan\left(\frac{y}{x}\right)=-\log|x|+C \qquad \text{...(2)}$$

When $x=1$, then $y=\frac{\pi}{4}$

$$\therefore \quad \tan\left(\frac{\pi}{4}\right)=-\log(1)+C$$
$$\Rightarrow \quad 1=-0+C \quad\Rightarrow\quad C=1$$

Put the value of 'C' in eq. (2), we get

$$\tan\left(\frac{y}{x}\right)=-\log|x|+1$$

which is the required particular solution.

24. Given differential equation can be rewritten as

$$\frac{dy}{dx}+\frac{y}{\cos^2 x}=\frac{\tan x}{\cos^2 x}$$

or $\frac{dy}{dx}+(\sec^2 x)y=\tan x\sec^2 x$

which is of the form $\frac{dy}{dx}+Py=Q$.

Here $P=\sec^2 x$ and $Q=\tan x\sec^2 x$

$$\therefore \quad \text{IF}=e^{\int P dx}=e^{\int\sec^2 x\,dx}=e^{\tan x}$$

$\therefore$ Complete solution is

$$y\cdot(\text{IF})=\int Q\cdot(\text{IF})\,dx\times C$$
$$\Rightarrow \quad y\times e^{\tan x}=\int e^{\tan x}\tan x\sec^2 x\,dx$$

Put $\tan x=t \Rightarrow \sec^2 x\,dx=dt$

$$\therefore \quad ye^{\tan x}=\int \underset{\text{II}}{e^t}\,\underset{\text{I}}{t}\,dt$$

TR!CK

Integration by parts,

$$\int \underset{\text{I}}{u}\,\underset{\text{II}}{v}\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx,$$

where u and v are the functions of x.

$$\Rightarrow \quad ye^{\tan x} = t\int e^t\,dt - \int\left[\frac{d}{dt}(t)\int e^t\,dt\right]dt$$

$$\Rightarrow \quad ye^{\tan x} = te^t - \int 1\times e^t\,dt$$

$$\Rightarrow \quad ye^{\tan x} = te^t - e^t + C$$

$$\Rightarrow \quad ye^{\tan x} = \tan x\, e^{\tan x} - e^{\tan x} + C$$

$$\Rightarrow \quad y = \tan x - 1 + Ce^{-\tan x}$$

Long Answer Type Questions

1. Given differential equation is

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \sqrt{(1+x^2)+y^2(1+x^2)} = -xy\frac{dy}{dx}$$

$$\Rightarrow \quad \sqrt{(1+x^2)(1+y^2)} = -xy\frac{dy}{dx}$$

$$\Rightarrow \quad \sqrt{1+x^2}\sqrt{1+y^2} = -xy\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{y}{\sqrt{1+y^2}}dy = -\frac{\sqrt{1+x^2}}{x}dx$$

On integrating both sides, we get

$$\int\frac{y}{\sqrt{1+y^2}}dy = -\int\frac{\sqrt{1+x^2}}{x^2}\cdot x\,dx$$

On putting $1+y^2 = t$ and $1+x^2 = u^2$

$$\Rightarrow \quad 2y\,dy = dt \text{ and } 2x\,dx = 2u\,du$$

$$\Rightarrow \quad y\,dy = \frac{dt}{2} \text{ and } x\,dx = u\,du$$

$$\therefore \quad \frac{1}{2}\int t^{-1/2}dt = -\int\frac{u}{u^2-1}\cdot u\,du \qquad \text{[on integrating]}$$

$$\Rightarrow \quad \frac{1}{2}\int t^{-1/2}dt = -\int\frac{u^2}{u^2-1}du$$

$$\Rightarrow \quad \frac{1}{2}\frac{t^{1/2}}{\frac{1}{2}} = -\int\frac{(u^2-1+1)}{u^2-1}du$$

$$\Rightarrow \quad t^{1/2} = -\int\frac{u^2-1}{u^2-1}du - \int\frac{1}{u^2-1}du$$

$$\Rightarrow \quad \sqrt{1+y^2} = -\int du - \int\frac{1}{u^2-(1)^2}du$$

[put $1+y^2 = t$]

$$\Rightarrow \quad \sqrt{1+y^2} = -u - \frac{1}{2}\log\left|\frac{u-1}{u+1}\right| + C$$

$$\left[\because \int\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$\therefore \quad \sqrt{1+y^2} = -\sqrt{1+x^2} - \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + C$$

which is the required solution.

2. Given, $\frac{d^2y}{dx^2} = x\sin x + e^x$

On integrating both sides with respect to x.

$$\frac{dy}{dx} = \int x\sin x\,dx + \int e^x$$

TR!CK

Integration by parts,

$$\int \underset{\text{I}}{u}\,\underset{\text{II}}{v}\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx,$$

where u and v are the functions of x.

$$\Rightarrow \frac{dy}{dx} = x\int\sin x\,dx - \int\left\{\frac{d}{dx}(x)\int\sin x\,dx\right\}dx + e^x$$

$$= -x\cos x + \int\cos x\,dx + e^x + C$$

$$= -x\cos x + \sin x + e^x + C \qquad \text{...(1)}$$

Again integrating both sides of eq. (1) w.r.t. 'x',

$$y = -\int x\cos x\,dx + \int\sin x\,dx + \int e^x + C\int 1\,dx + d$$

$$= -x\cdot\int\cos x\,dx + \int\left\{\frac{d}{dx}(x)\int\cos x\,dx\right\}dx$$

$$-\cos x + e^x + Cx + d$$

$$= -x\sin x + \int\sin x\,dx - \cos x + e^x + Cx + d$$

$$= -x\sin x - \cos x - \cos x + e^x + Cx + d$$

$$\therefore \quad y = -x\sin x - 2\cos x + e^x + Cx + d$$

which is the required solution.

3. Given that, $y' = e^x\sin x$

$$\Rightarrow \quad \frac{dy}{dx} = e^x\sin x$$

$$\Rightarrow \quad dy = e^x\sin x\,dx$$

$$\therefore \quad \int dy = \int e^x\sin x\,dx \qquad \text{[on integrating]}$$

$$\Rightarrow \quad y = \int e^x\sin x\,dx + C \qquad \text{...(1)}$$

Let $I = \int \underset{\text{I}}{e^x}\,\underset{\text{II}}{\sin x}\,dx$ [use integration by parts]

TR!CK

Integration by parts,

$$\int \underset{\text{I}}{u}\,\underset{\text{II}}{v}\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx,$$

where u and v are some functions of x.

$$\Rightarrow I = e^x\int\sin x\,dx - \int\left\{\frac{d}{dx}e^x\int\sin x\,dx\right\}dx$$

$$= e^x(-\cos x) - \int e^x(-\cos x)\,dx$$

$$= -e^x\cos x + \left[e^x\int\cos x\,dx - \int\left\{\frac{d}{dx}e^x\int\cos x\,dx\right\}dx\right]$$

$$= -e^x\cos x + (e^x\sin x - \int e^x\sin x\,dx)$$

$$= -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx$$

$$\Rightarrow \quad I = e^x(\sin x - \cos x) - I$$

$$\Rightarrow \quad 2I = e^x(\sin x - \cos x)$$

$$\Rightarrow \quad I = \frac{e^x}{2}(\sin x - \cos x) + C$$

From eq. (1), we get

$$y = \frac{e^x}{2}(\sin x - \cos x) + C \qquad \text{...(2)}$$

Put $x = 0$ and $y = 0$, we get

$$0 = \frac{e^0}{2}(\sin 0 - \cos 0) + C$$

$$\Rightarrow \quad 0 = \frac{1}{2}(0-1) + C \Rightarrow C = \frac{1}{2}$$

Put the value of C in eq. (2), we get

$$y = \frac{1}{2}e^x(\sin x - \cos x) + \frac{1}{2}$$

$$\therefore \quad 2y - 1 = e^x(\sin x - \cos x)$$

which is the required equation of curve.

4. Given differential equation is

$$x^2y\,dx - (x^3+y^3)\,dy = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x^3+y^3}{x^2y} \quad \text{...(1)}$$

which is a homogeneous differential equation.

So put $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ in eq. (1), we get

$$v + y\frac{dv}{dy} = \frac{y^3+(vy)^3}{(vy)^2\cdot y} = \frac{y^3+v^3y^3}{v^2y^3}$$

$$\Rightarrow \quad v + y\frac{dv}{dy} = \frac{1+v^3}{v^2}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{1+v^3}{v^2} - v = \frac{v^3-v^3+1}{v^2}$$

$$\Rightarrow \quad \frac{dy}{y} = v^2\,dv$$

On integrating, we have

$$\int\frac{dy}{y} = \int v^2dv \Rightarrow \log|y| = \frac{v^3}{3} + C$$

$$\Rightarrow \quad \log|y| = \frac{x^3}{3y^3} + C \quad [\because x = vy]$$

$$\Rightarrow \quad \log|y| - \frac{x^3}{3y^3} = C$$

COMMON ERR(!)R

Some students put wrong substitution i.e., $y = vx$ in the homogeneous equation of the form $\frac{dx}{dy} = f(x, y)$.

5. Given differential equation is

$$(x^2-y^2)\,dx + 2xy\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2-x^2}{2xy} \quad \text{...(1)}$$

which is a homogeneous differential equation.

TiP

Students remember that in homogeneous equation of the types $\frac{dy}{dx} = f(x, y)$ or $\frac{dx}{dy} = f(x, y)$, put $y = vx$ or $x = vy$ respectively.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq. (1) we get

$$v + x\frac{dv}{dx} = \frac{(vx)^2-x^2}{2x(vx)} = \frac{x^2(v^2-1)}{2x^2v}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2-1}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2-1}{2v} - v = \frac{v^2-1-2v^2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-1-v^2}{2v}$$

Separating the variables, we get

$$\frac{2v}{-1-v^2}dv = \frac{dx}{x} \Rightarrow \frac{2v}{1+v^2}dv = -\frac{dx}{x}$$

On integrating, we get

$$\int\frac{2v}{1+v^2}dv = -\int\frac{dx}{x}$$

TR!CK

$$\int\frac{f'(x)}{f(x)}dx = \log|f(x)| + C$$

$$\Rightarrow \quad \log|1+v^2| = -\log|x| + \log|C|$$

$$\Rightarrow \quad \log|1+v^2| + \log|x| = \log|C|$$

$$\Rightarrow \quad \log|(1+v^2)\cdot x| = \log|C|$$

$$\Rightarrow \quad x(1+v^2) = C$$

$$\Rightarrow \quad x\left(1+\frac{y^2}{x^2}\right) = C \quad [\because y = vx]$$

$$\Rightarrow \quad x\left(\frac{x^2+y^2}{x^2}\right) = C$$

$$\Rightarrow \quad x^2+y^2 = Cx$$

6. Given differential equation is

$$(x^3-3xy^2)\,dx = (y^3-3x^2y)\,dy$$

TiP

Learn how to distinguish between variable separable, homogeneous and linear differential equations.

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^3-3xy^2}{y^3-3x^2y} \quad \text{...(1)}$$

which is a homogeneous differential equation.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq. (1), we get

$$v + x\frac{dv}{dx} = \frac{x^3-3x(vx)^2}{(vx)^3-3x^2(vx)}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{x^3-3v^2x^3}{v^3x^3-3vx^3} = \frac{x^3(1-3v^2)}{x^3(v^3-3v)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v} - v = \frac{1-3v^2-v^4+3v^2}{v^3-3v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1-v^4}{v^3-3v} \Rightarrow \frac{v^3-3v}{1-v^4}dv = \frac{dx}{x}$$

$$\Rightarrow \quad \left\{\frac{v^3}{1-v^4} - \frac{3v}{1-v^4}\right\}dv = \frac{dx}{x}$$

$$\Rightarrow \int\frac{v^3}{1-v^4}dv - 3\int\frac{v}{1-v^4}dv = \int\frac{dx}{x} \quad \text{[on integrating]}$$

Let $\quad t = 1-v^4 \Rightarrow dt = -4v^3\,dv$

TR!CK

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$$

$$\Rightarrow \quad -\frac{1}{4}\int \frac{dt}{t} - 3\int \frac{v\,dv}{1-v^4} = \int \frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{4}\log|t| - 3\int \frac{v}{1-v^4}dv = \log|x| + C_1$$

$$\Rightarrow -\frac{1}{4}\log|1-v^4| - 3\int \frac{v}{1-v^4}dv = \log|x| + C_1 \quad ...(1)$$

By partial fraction,

$$\frac{v}{(1-v^4)} = \frac{v}{(1-v^2)(1+v^2)} = \frac{Av+B}{1-v^2} + \frac{Cv+D}{1+v^2}$$

$$\Rightarrow \quad v = (Av+B)(1+v^2) + (Cv+D)(1-v^2)$$

$$\Rightarrow \quad v = Av + B + Av^3 + Bv^2 + Cv + D - Cv^3 - Dv^2$$

$$\Rightarrow \quad v = (B+D) + (A+C)v + (B-D)v^2 + (A-C)v^3$$

On comparing the like powers of v, we get

$$B + D = 0 \Rightarrow B = -D \quad ...(2)$$

$$A + C = 1 \quad ...(3)$$

$$B - D = 0 \Rightarrow B = D \quad ...(4)$$

and $\quad A - C = 0 \Rightarrow A = C \quad ...(5)$

From eqs. (2) and (4), we get

$$B = D = 0$$

From eqs. (3) and (5), we get

$$2C = 1 \Rightarrow C = \frac{1}{2} \text{ and } A = \frac{1}{2}$$

$$\therefore \quad \frac{v}{1-v^4} = \frac{\frac{1}{2}v + 0}{1-v^2} + \frac{\frac{1}{2}v + 0}{1+v^2}$$

From eq. (1), we get

$$-\frac{1}{4}\log|1-v^4| - 3\int\left\{\frac{v}{2(1-v^2)} + \frac{v}{2(1+v^2)}\right\}dv = \log|x| + C_1$$

$$\Rightarrow \quad -\frac{1}{4}\log|1-v^4| + \frac{3}{4}\int\frac{-2v}{1-v^2}dv - \frac{3}{4}\int\frac{2v}{1+v^2}dv = \log|x| + C_1$$

TR!CK

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$$

$$\Rightarrow \quad -\frac{1}{4}\log|1-v^4| + \frac{3}{4}\log|1-v^2| - \frac{3}{4}\log|1+v^2| = \log|x| + C_1$$

$$\Rightarrow \quad \frac{3}{4}\log\left|\frac{1-v^2}{1+v^2}\right| = \frac{1}{4}\log|1-v^4| + \log|x| + C_1$$

$$\Rightarrow \frac{3}{4}\log\left|\frac{1-y^2/x^2}{1+y^2/x^2}\right| = \frac{1}{4}\log|1-y^4/x^4| + \log|x| + C_1$$

$[\because y = vx]$

$$\Rightarrow \frac{3}{4}\log\left|\frac{x^2-y^2}{x^2+y^2}\right| = \frac{1}{4}\log\left|\frac{x^4-y^4}{x^4}\right| + \log|x| + C_1$$

$$\Rightarrow \quad \frac{3}{4}\log\left|\frac{x^2-y^2}{x^2+y^2}\right| = \frac{1}{4}\log|x^4-y^4| - \frac{1}{4}\log x^4 + \log|x| + C_1$$

$$\Rightarrow \quad \frac{3}{4}\log\left|\frac{x^2-y^2}{x^2+y^2}\right| = \frac{1}{4}\log|x^4-y^4| - \log|x| + \log|x| + C_1$$

$$\Rightarrow \quad 3\log\left|\frac{x^2-y^2}{x^2+y^2}\right| = \log|x^4-y^4| + 4C_1$$

$$\Rightarrow \quad \log\left|\frac{(x^2-y^2)^3}{(x^2+y^2)^3}\right| - \log|(x^2+y^2)(x^2-y^2)| = 4C_1$$

$$\Rightarrow \quad \log\left|\frac{(x^2-y^2)^3}{(x^2+y^2)^3(x^2+y^2)(x^2-y^2)}\right| = 4C_1$$

$$\Rightarrow \quad \log\left|\frac{(x^2-y^2)^2}{(x^2+y^2)^4}\right| = 4C_1$$

$$\Rightarrow \quad \log\left|\frac{(x^2-y^2)^2}{(x^2+y^2)^4}\right| = \log C^2 \quad [\text{where, } 4C_1 = \log C^2]$$

$$\Rightarrow \quad \frac{(x^2-y^2)^2}{(x^2+y^2)^4} = C^2 \Rightarrow (x^2-y^2)^2 = C^2(x^2+y^2)^4$$

$$\therefore \quad (x^2-y^2) = C(x^2+y^2)^2$$

[taking square root on both sides] **Hence proved.**

7. Given,

$$(xdy - ydx)\,y\sin\left(\frac{y}{x}\right) = (ydx + xdy)\,x\cos\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad xy\sin\left(\frac{y}{x}\right)dy - y^2\sin\left(\frac{y}{x}\right)dx = xy\cos\left(\frac{y}{x}\right)dx + x^2\cos\left(\frac{y}{x}\right)dy$$

TiP

In homogeneous equation, the degree of all terms will be the same.

$$\Rightarrow \quad \left\{xy\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)\right\}dy = \left\{xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)\right\}dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}}{x\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}} \quad ...(1)$$

which is a homogeneous differential equation.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq. (1), we get

$$\frac{dy}{dx} = \frac{vx\left\{x\cos\left(\frac{vx}{x}\right) + vx\sin\left(\frac{vx}{x}\right)\right\}}{x\left\{vx\sin\left(\frac{vx}{x}\right) - x\cos\left(\frac{vx}{x}\right)\right\}}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v(\cos v + v\sin v)}{v\sin v - \cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \quad \frac{v\sin v - \cos v}{2v\cos v}dv = \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{v\sin v - \cos v}{v\cos v}dv = 2\int \frac{dx}{x}$$

[on integrating]

$$\Rightarrow \quad \int\left(\tan v - \frac{1}{v}\right)dv = 2\int\frac{dx}{x}$$

TR!CK

$\int \tan ax\, dx = \frac{1}{a}\log|\sec ax| + C$

$$\Rightarrow \quad \log|\sec v| - \log|v| = 2\log|x| + \log C$$

$$\Rightarrow \quad \log\left|\frac{\sec v}{v}\right| - \log x^2 = \log C$$

$$\Rightarrow \quad \log\left|\frac{\sec v}{vx^2}\right| = \log C$$

$$\Rightarrow \quad \frac{\sec v}{vx^2} = C$$

$$\Rightarrow \quad \frac{\sec\left(\frac{y}{x}\right)}{\frac{y}{x}\times x^2} = C \qquad [\because y = vx]$$

$$\Rightarrow \quad \sec\left(\frac{y}{x}\right) = Cxy$$

which is the required solution.

8. Given, $x^2dy + (xy + y^2)\,dx = 0$

$$\Rightarrow \quad x^2dy = -(xy+y^2)\,dx$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(xy+y^2)}{x^2} \qquad ...(1)$$

TiP

In homogeneous equations, the degree of all terms will be the same.

$\because$ $(xy+y^2)$ and x^2 each are homogeneous function of degree 2.

$\therefore$ Eq. (1) is a homogeneous differential equation.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq. (1), we get

$$v + x\frac{dv}{dx} = -\frac{(x\cdot vx + v^2x^2)}{x^2} = -(v+v^2)$$

$$\Rightarrow \quad x\frac{dv}{dx} = -v - v - v^2 = -2v - v^2 = -v(2+v)$$

Separate the variables,

$$-\frac{dx}{x} = \frac{dv}{v(2+v)}$$

$$\Rightarrow \quad -\frac{dx}{x} = \frac{1}{2}\left[\frac{1}{v} - \frac{1}{v+2}\right]dv$$

[by partial fraction]

$$\Rightarrow \quad -\int\frac{dx}{x} = \frac{1}{2}\int\left(\frac{1}{v} - \frac{1}{v+2}\right)dv \text{ [on integrating]}$$

$$\Rightarrow \quad -\log|x| = \frac{1}{2}[(\log v - \log|v+2|) + C$$

$$\Rightarrow \quad -\log|x| = \frac{1}{2}\log\left|\frac{v}{v+2}\right| + C \qquad \left[\because v = \frac{y}{x}\right]$$

$$\Rightarrow \quad -\log|x| = \frac{1}{2}\log\left|\frac{\frac{y}{x}}{\frac{y}{x}+2}\right| + C$$

$$\Rightarrow \quad -\log|x| = \frac{1}{2}\log\left|\frac{y}{y+2x}\right| + C \qquad ...(2)$$

Given, when $x = 1$, then $y = 1$

$$-\log(1) = \frac{1}{2}\log\left(\frac{1}{1+2}\right) + C$$

$$\Rightarrow \quad 0 = \frac{1}{2}\log\left(\frac{1}{3}\right) + C$$

$$\Rightarrow \quad C = -\frac{1}{2}\log\left(\frac{1}{3}\right)$$

Put the value of C in eq. (2),

$$-\log|x| = -\frac{1}{2}\left[\log\left(\frac{1}{3}\right) - \log\left|\frac{y}{y+2x}\right|\right]$$

$$\Rightarrow \quad 2\log|x| = \log\left|\frac{y+2x}{3y}\right|$$

$$\Rightarrow \quad \log x^2 = \log\left(\frac{y+2x}{3y}\right)$$

$$\Rightarrow \quad x^2 = \frac{y+2x}{3y} \Rightarrow 3x^2y = y + 2x$$

9. Given differential equation is

$$(x-y)\frac{dy}{dx} = (x+2y) \Rightarrow \frac{dy}{dx} = \frac{x+2y}{x-y} \qquad ...(1)$$

TiP

In homogeneous equation, the degree of all terms will be the same.

$\because$ $(x+2y)$ and $(x-y)$ each are homogeneous function of degree one.

$\therefore$ Eq. (1) is a homogeneous differential equation.

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dy}{dx}$ in eq. (1), we get

$$v + x\frac{dv}{dx} = \frac{x+2vx}{x-vx} = \frac{1+2v}{1-v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

$$= \frac{1+2v-v+v^2}{1-v}$$

$$= \frac{1+v+v^2}{1-v}$$

$$\Rightarrow \quad \frac{-1+v}{1+v+v^2}dv = \frac{-dx}{x}$$

$$\Rightarrow \quad \int\frac{v-1}{v^2+v+1}dv = -\int\frac{dx}{x} \qquad \text{[on integrating]}$$

$$\Rightarrow \quad \frac{1}{2}\int\frac{2v+1-3}{v^2+v+1}dv = -\log|x| + C$$

$$\Rightarrow \quad \frac{1}{2}\int\frac{2v+1}{v^2+v+1}dv - \frac{3}{2}\int\frac{dv}{v^2+v+1} = -\log|x| + C$$

TR!CK

$\int\frac{f'(x)}{f(x)}dx = \log|f(x)| + C$

$$\Rightarrow \frac{1}{2}\log|v^2+v+1| - \frac{3}{2}\int \frac{dv}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = -\log|x| + C$$

TR!CK

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\Rightarrow \frac{1}{2}\log|v^2+v+1| - \frac{3}{2}\cdot\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left\{\frac{\left(v+\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}\right\} = -\log|x| + C$$

$$\Rightarrow \frac{1}{2}\log|v^2+v+1| - \sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = -\log|x| + C$$

$$\Rightarrow \frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right| - \sqrt{3}\tan^{-1}\left(\frac{2y+x}{x\sqrt{3}}\right) = -\log|x| + C \qquad \left[\because v = \frac{y}{x}\right]$$

$$\Rightarrow \frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right| + \frac{1}{2}\log x^2 = \sqrt{3}\tan^{-1}\left(\frac{2y+x}{x\sqrt{3}}\right) + C$$

$$\Rightarrow \frac{1}{2}\log\left|\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right)x^2\right| = \sqrt{3}\tan^{-1}\left(\frac{2y+x}{x\sqrt{3}}\right) + C$$

$$\Rightarrow \log|y^2+xy+x^2| = 2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{x\sqrt{3}}\right) + 2C \qquad \ldots(2)$$

Now, putting $y = 0$ and $x = 1$,

$$\log|0+0+1| = 2\sqrt{3}\tan^{-1}\left(\frac{0+1}{\sqrt{3}}\right) + 2C$$

$$\Rightarrow \log(1) = 2\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + 2C$$

$$\Rightarrow 0 = 2\sqrt{3}\times\frac{\pi}{6} + 2C \Rightarrow 2C = -\frac{\pi}{\sqrt{3}}$$

Put the value of $2C$ in eq. (2), we get

$$\log|x^2+xy+y^2| = 2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{x\sqrt{3}}\right) - \frac{\pi}{\sqrt{3}}$$

$$\Rightarrow \sqrt{3}\log|x^2+xy+y^2| = 6\tan^{-1}\left(\frac{2y+x}{x\sqrt{3}}\right) - \pi$$

$$\therefore \quad 6\tan^{-1}\left(\frac{2y+x}{x\sqrt{3}}\right) = \sqrt{3}\log|x^2+xy+y^2| + \pi$$

10. Given differential equation is

$$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$$

$$\Rightarrow x\,dy = (y+\sqrt{x^2+y^2})\,dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \qquad \ldots(1)$$

which is a homogeneous differential equation.

Now put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in eq. (1), we get

$$v + x\frac{dv}{dx} = \frac{vx+\sqrt{x^2+v^2x^2}}{x}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{vx+x\sqrt{1+v^2}}{x} - v$$

$$\Rightarrow x\frac{dv}{dx} = v + \sqrt{1+v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2} \Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

$$\Rightarrow \int\frac{dv}{\sqrt{1+v^2}} = \int\frac{dx}{x} \qquad \text{[on integrating]}$$

TR!CK

$$\int\frac{dx}{\sqrt{a^2+x^2}} = \log|x+\sqrt{a^2+x^2}| + C$$

$$\Rightarrow \log|v+\sqrt{1+v^2}| = \log|x| + C$$

$$\Rightarrow \log\left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right| = \log|x| + C \qquad \left[\because v = \frac{y}{x}\right]$$

$$\Rightarrow \log|y+\sqrt{x^2+y^2}| - \log|x| = \log|x| + C$$

$$\Rightarrow \log|y+\sqrt{x^2+y^2}| = 2\log|x| + C \qquad \ldots(2)$$

Now, putting $y = 0$ and $x = 1$ in eq. (2), we get

$$\log|0+\sqrt{(1)^2+0}| = 2\log(1) + C$$

$$\Rightarrow \log 1 = 2\log(1) + C$$

$$\Rightarrow 0 = 2\times 0 + C \Rightarrow C = 0$$

Put the value of C in eq. (2), we get

$$\log|y+\sqrt{x^2+y^2}| = 2\log|x| + 0$$

$$\Rightarrow \log|y+\sqrt{x^2+y^2}| = \log x^2$$

$$\Rightarrow y + \sqrt{x^2+y^2} = x^2$$

which is the required solution.

COMMON ERROR

Students could not recognize the form of differential equation correctly.

11. Given, $ye^y dx = (y^3+2xe^y)\,dy$

$$\Rightarrow \frac{dx}{dy} = \frac{y^3+2xe^y}{ye^y} = \frac{y^2}{e^y} + \frac{2x}{y}$$

TIP

Practice more problems based on finding particular solution.

$$\Rightarrow \frac{dx}{dy} - \frac{2}{y}\cdot x = \frac{y^2}{e^y}$$

On comparing with $\frac{dx}{dy} + Px = Q$, we get

$$P = -\frac{2}{y} \text{ and } Q = \frac{y^2}{e^y}$$

$$\text{IF} = e^{\int P\,dy} = e^{\int -\frac{2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$$

$\therefore$ General solution is

$$x(\text{IF}) = \int Q\cdot(\text{IF})\,dy + C$$

$$\Rightarrow x\cdot\frac{1}{y^2} = \int\frac{y^2}{e^y}\times\frac{1}{y^2}\,dy + C$$

$$\Rightarrow \frac{x}{y^2} = \int e^{-y}dy + C \Rightarrow \frac{x}{y^2} = -e^{-y} + C \qquad \ldots(1)$$

Given, $x = 0$ and $y = 1$

From eq. (1),

$$\frac{0}{1} = -e^{-1} + C$$

$$\Rightarrow \quad 0 = -\frac{1}{e} + C \Rightarrow C = \frac{1}{e}$$

Put the value of C in eq. (1),

$$\frac{x}{y^2} = -e^{-y} + \frac{1}{e}$$

Hence, required solution is

$$x = -e^{-y}y^2 + e^{-1}y^2$$

12. Given equation is

$$x\frac{dy}{dx} + y = x^3y^6 \Rightarrow \frac{1}{y^6}\frac{dy}{dx} + \frac{1}{xy^5} = x^2$$

TiP

The equation $\frac{dy}{dx} + Py = Qy^n$ or $\frac{dx}{dy} + Px = Qx^n$ is known as Bernoulli's equation.

Let $\frac{1}{y^5} = z$

$$\Rightarrow \quad -\frac{5}{y^6}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \quad \frac{1}{y^6}\frac{dy}{dx} = -\frac{dz}{5\,dx}$$

$$\therefore \quad -\frac{dz}{5\,dx} + \frac{z}{x} = x^2$$

$$\Rightarrow \quad \frac{dz}{dx} - \frac{5}{x}z = -5x^2$$

Comparing with $\frac{dz}{dx} + Pz = Q$

Here $P = -\frac{5}{x}$, and $Q = -5x^2$

$$\therefore \quad \text{IF} = e^{\int P\,dx} = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5}$$

Now solution is

$$z \cdot x^{-5} = \int -5x^2 \cdot x^{-5}dx + C = -5\int x^{-3}dx + C$$

$$= -5 \cdot \frac{x^{-2}}{-2} + C = \frac{5}{2x^2} + C$$

$$\Rightarrow \quad z = \frac{5}{2}x^3 + Cx^5 \Rightarrow \frac{1}{y^5} = \frac{5}{2}x^3 + Cx^5$$

13. Given, $\frac{dy}{dx} + y\tan x = y^2\sec x$

TiP

The equation $\frac{dy}{dx} + Py = Qy^n$ or $\frac{dx}{dy} + Px = Qx^n$ is known as Bernoulli's equation.

Dividing on both sides by y^2,

$$\frac{1}{y^2}\cdot\frac{dy}{dx} + \frac{1}{y}\cdot\tan x = \sec x \quad \text{...(1)}$$

Let $\frac{1}{y} = t$

On differentiating both sides w.r.t. 'x',

$$\frac{-1}{y^2}\cdot\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{1}{y^2}\cdot\frac{dy}{dx} = \frac{-dt}{dx}$$

From eq. (1), $\frac{-dt}{dx} + t\cdot\tan x = \sec x$

$$\Rightarrow \quad \frac{dt}{dx} - \tan x \cdot t = -\sec x$$

On comparing with $\frac{dt}{dx} + P\cdot t = Q$, we get

$$P = -\tan x \quad \text{and} \quad Q = -\sec x$$

$$\therefore \quad \text{IF} = e^{\int P\,dx} = e^{\int -\tan x\,dx} = e^{-\log|\sec x|}$$

$$= e^{\log\left|\frac{1}{\sec x}\right|} = e^{\log|\cos x|} = \cos x$$

So, required complete solution is

$$t\cdot(\text{IF}) = \int (\text{IF})\cdot Q\,dx + C$$

$$\Rightarrow \quad t\cdot\cos x = \int \cos x \times (-\sec x)\,dx + C$$

$$\Rightarrow \quad t\cdot\cos x = -\int \cos x \cdot \frac{1}{\cos x}dx + C = -\int 1\,dx + C$$

$$\Rightarrow \quad \frac{1}{y}\cdot\cos x = -x + C \qquad \left[\because t = \frac{1}{y}\right]$$

$$\therefore \quad \cos x = -xy + Cy$$

which is the required solution.

14. Given differential equation is

$$(1+y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (e^{\tan^{-1}y} - x)\frac{dy}{dx} = (1+y^2)$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{(e^{\tan^{-1}y} - x)}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{e^{\tan^{-1}y}}{1+y^2} - \frac{x}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2}$$

On comparing with $\frac{dx}{dy} + Px = Q$, we get

$$P = \frac{1}{1+y^2} \quad \text{and} \quad Q = \frac{e^{\tan^{-1}y}}{1+y^2}$$

$$\therefore \quad \text{IF} = e^{\int P\,dy} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}$$

So, general solution is,

$$x\cdot(\text{IF}) = \int Q\cdot(\text{IF})\,dy + C$$

$$\Rightarrow \quad x\cdot e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2} \times e^{\tan^{-1}y}dy + C$$

TR!CK

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Put $t = \tan^{-1}y$

$$\Rightarrow \quad dt = \frac{dy}{1+y^2}$$

$$\therefore \quad xe^{\tan^{-1}y} = \int e^t\cdot e^t dt + C = \int e^{2t}dt + C$$

$$\Rightarrow \quad xe^{\tan^{-1}y} = \frac{1}{2}e^{2t} + C$$

$$\Rightarrow \quad xe^{\tan^{-1}y} = \frac{1}{2}e^{2\tan^{-1}y} + C$$

$$\Rightarrow \quad x = \frac{1}{2}e^{\tan^{-1}y} + Ce^{-\tan^{-1}y} \quad \text{...(1)}$$

Put $y = 0$ and $x = 1$ in eq. (1), we get

$$1 = \frac{1}{2}e^{\tan^{-1}0} + Ce^{-\tan^{-1}0}$$

$$\Rightarrow \quad 1 = \frac{1}{2}e^{0} + C\cdot e^{0}$$

$$\Rightarrow \quad 1 = \frac{1}{2}\times 1 + C\times 1 \;\Rightarrow\; C = 1 - \frac{1}{2} = \frac{1}{2}$$

Put the value of C in eq. (1), we get

$$x = \frac{1}{2}e^{\tan^{-1}y} + \frac{1}{2}e^{-\tan^{-1}y}$$

$$\therefore \quad 2x = e^{\tan^{-1}y} + e^{-\tan^{-1}y}$$

which is the required particular solution.

COMMON ERR(!)R

Students forget to find the particular solution after finding the general solution.

15. Given that, $\frac{dy}{dx} + \cot x \cdot y = \frac{2}{1+\sin x}$

On comparing with $\frac{dy}{dx} + Py = Q$, we get

$$P = \cot x \text{ and } Q = \frac{2}{1+\sin x}$$

$$\therefore \quad \text{IF} = e^{\int P\,dx} = e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$$

So, complete solution is

$$y\cdot(\text{IF}) = \int Q\cdot(\text{IF})\,dx + C$$

$$\Rightarrow y\cdot\sin x = \int \frac{2}{1+\sin x}\times \sin x\,dx + C$$

$$= \int\left\{\frac{2(\sin x+1)}{(1+\sin x)} - \frac{2}{1+\sin x}\right\}dx + C$$

$$= 2\int dx - 2\int \frac{dx}{1+\sin x} + C$$

TR!CK

$$\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$$

$$= 2x - 2\int \frac{1+\tan^2 x/2}{1+\tan^2\frac{x}{2} + 2\tan\frac{x}{2}}dx + C$$

$$= 2x - 2\int \frac{\sec^2\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2}dx + C$$

$$\left[\text{put } t = 1+\tan\frac{x}{2} \Rightarrow dt = \frac{1}{2}\sec^2\frac{x}{2}dx\right]$$

$$= 2x - 2\int \frac{2dt}{t^2} + C$$

$$\Rightarrow y\cdot\sin x = 2x + 4\cdot\frac{1}{t} + C = 2x + \frac{4}{\left(1+\tan\frac{x}{2}\right)} + C \quad \text{...(1)}$$

When $x = \frac{\pi}{4}$, then $y = 0$

$$\therefore \quad 0 = 2\cdot\frac{\pi}{4} + \frac{4}{1+\tan\frac{\pi}{8}} + C$$

$$\Rightarrow \quad C = -\left(\frac{\pi}{2} + \frac{4}{1+\tan\frac{\pi}{8}}\right)$$

Put the value of C in eq. (1), we get

$$y\cdot\sin x = 2x + \frac{4}{1+\tan\frac{x}{2}} - \left(\frac{\pi}{2} + \frac{4}{1+\tan\frac{\pi}{8}}\right)$$

$$\Rightarrow \; y = \operatorname{cosec} x\left[2\left\{x + \frac{2}{1+\tan\frac{x}{2}}\right\} - \left(\frac{\pi}{2} + \frac{4}{1+\tan\frac{\pi}{8}}\right)\right].$$

which is the required particular solution.

16. Given, $\frac{dy}{dx} - 3y\cot x = \sin 2x$

which is a differential equation of the form $\frac{dy}{dx} + Py = Q$.

Here, $P = -3\cot x$ and $Q = \sin 2x$

TR!CK

$$\int \cot ax\,dx = \frac{1}{a}\log|\sin ax| + C$$

$$\therefore \; \text{IF} = e^{\int P\,dx} = e^{-\int 3\cot x\,dx} = e^{-3\log \sin x} = e^{\log\frac{1}{\sin^3 x}}$$

$$= \frac{1}{\sin^3 x} = \operatorname{cosec}^3 x$$

Now, complete general solution is

$$y\operatorname{cosec}^3 x = \int \sin 2x\cdot\operatorname{cosec}^3 x\,dx + C$$

$$\Rightarrow \quad y\operatorname{cosec}^3 x = \int 2\sin x\cos x\times\frac{1}{\sin^3 x}dx + C$$

$$\Rightarrow \quad y\operatorname{cosec}^3 x = 2\int \operatorname{cosec} x\cot x\,dx + C$$

$$= -2\operatorname{cosec} x + C \quad \text{...(1)}$$

Put $x = \frac{\pi}{2}, y = 2$ in the eq. (1), we get

$$2\times\operatorname{cosec}^3\frac{\pi}{2} = -2\operatorname{cosec}\frac{\pi}{2} + C$$

$$\Rightarrow \quad 2\times 1^3 = -2\times 1 + C$$

$$\Rightarrow \quad C = 2 + 2 = 4$$

Put $C = 4$ in eq. (1), we get

$$y\operatorname{cosec}^3 x = -2\operatorname{cosec} x + 4$$

$$\Rightarrow \quad y = -2\sin^2 x + 4\sin^3 x$$

which is the required particular solution.

17. Given differential equation is

$$x\frac{dy}{dx} + y = x\cos x + \sin x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{x}\cdot y = \frac{x\cos x + \sin x}{x}$$

On comparing with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{1}{x} \text{ and } Q = \frac{x\cos x + \sin x}{x}$$

$$\therefore \quad \text{IF} = e^{\int P\,dx} = e^{\int\frac{1}{x}dx} = e^{\log|x|} = x$$

So, complete general solution is

$$y\cdot(\text{IF}) = \int Q\cdot(\text{IF})\,dx + C$$

$$\Rightarrow \quad yx = \int \frac{x\cos x + \sin x}{x}\times x\,dx + C$$

$$\Rightarrow \quad yx = \int (x\cos x + \sin x)\,dx + C$$

$$\Rightarrow \quad yx = \int (x\cos x\,dx + \int \sin x\,dx + C$$

$$\Rightarrow \quad yx = x\cdot\int\cos x\,dx - \int\left\{\frac{d}{dx}x\int\cos x\,dx\right\}dx + \int \sin x\,dx + C$$

$\Rightarrow \quad yx = x\sin x - \int 1\cdot\sin x\,dx - \cos x + C$

$\Rightarrow \quad yx = x\sin x - (-\cos x) - \cos x + C$

$\Rightarrow \quad xy = x\sin x + \cos x - \cos x + C$

$\Rightarrow \quad xy = x\sin x + C \qquad ...(1)$

TR!CK

Integration by parts,

$$\int \underset{\text{I}}{u}\cdot\underset{\text{II}}{v}\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx,$$

where u and v are the functions of x.

Put $y = 1$ and $x = \frac{\pi}{2}$ in eq. (1), we get

$\therefore \quad \frac{\pi}{2}(1) = \frac{\pi}{2}\sin\frac{\pi}{2} + C$

$\Rightarrow \quad \frac{\pi}{2} = \frac{\pi}{2}(1) + C$

$\Rightarrow \quad C = 0$

From eq. (1), we get

$xy = x\sin x \quad \Rightarrow \quad y = \sin x$

which is the required particular solution.

Chapter Test

Multiple Choice Questions

Q 1. The general solution of $\frac{dy}{dx} = \sqrt{4-y^2}$, where $-2 < y < 2$, is:

a. $y = \sin(x+C)$ b. $y = 2\sin(x+C)$

c. $y = 2\cos(x+C)$ d. $y = \cos(x+C)$

Q 2. The integrating factor of $\frac{dy}{dx} + y = \frac{1+y}{x}$ is:

a. $\frac{e^x}{x}$ b. $\frac{e^{-x}}{x}$ c. e^x d. e^{-x}

Assertion and Reasons Type Questions

Directions (Q. Nos. 3-4): *In the following questions, each question contains Assertion (A) and Reason (R). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are:*

a. Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A)

b. Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)

c. Assertion (A) is true but Reason (R) is false

d. Assertion (A) is false but Reason (R) is true

Q 3. Assertion (A): $F(x, y) = \frac{\sqrt{x^2+y^2}+y}{x}$ is a homogeneous function of degree zero.

Reason (R): A differential equation of the form $\frac{dy}{dx} = F(x, y)$ is said to be homogeneous, if $F(x, y)$ is a homogeneous function of degree zero, whereas a function $F(x, y)$ is a homogeneous function of degree n, if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$.

Q 4. Assertion (A): $\frac{dy}{dx} + y = 10$ is a differential equation of the type $\frac{dy}{dx} + Py = Q$ but it can be solved using variable separable method also.

Reason (R): Integrating factor of the differential equation of the form $\frac{dx}{dy} + P_1 x = Q_1$ is given by $e^{\int P_1\,dy}$.

Case Study Based Questions

Q 5. Case Study 1

If an equation is of the form $\frac{dy}{dx} + Py = Q$, where P, Q are functions of x, then such equation is known as linear differential equation. Its solution is given by $y\cdot(\text{IF}) = \int Q\cdot(\text{IF})\,dx + C$, where $\text{IF} = e^{\int P\,dx}$.

Now, suppose the given equation is $(1+\sin x)\frac{dy}{dx} + y\cos x + x = 0$.

Based on the above information, solve the following questions:

(i) Find the value of IF:

(ii) Find the solution of given differential equation.

(iii) If $y(0) = 1$, then find the value of y.

OR

Find the value of $y\left(\frac{\pi}{2}\right)$.

Q 6. Case Study 2

Order: The order of a differential equation is the order of the highest order derivative appearing in the differential equation.

Degree: The degree of differential equation is the power of the highest order derivative, when differential coefficients are made free from radicals and fractions. Also, differential equation must be a polynomial equation in derivatives for the degree to be defined.

Based on the given information, solve the following questions:

(i) Find the order and degree of the differential equation $y\frac{dy}{dx}=\frac{x}{\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^3}$.

(ii) Determine degree of the differential equation $(\sqrt{a+x})\cdot\left(\frac{dy}{dx}\right)+x=0$.

(iii) Find the order and degree of the differential equation $\left(1+\left(\frac{dy}{dx}\right)^3\right)^{\frac{7}{3}}=7\frac{d^2y}{dx^2}$.

Or

Find the difference of the order and degree of the differential equation

$$\left(1+\left(\frac{dy}{dx}\right)^3\right)^{\frac{7}{3}}=7\frac{d^2y}{dx^2}$$

Very Short Answer Type Questions

Q 7. Write the order and degree of the differential equation $\frac{d^2y}{dx^2}=x+\sqrt{\frac{dy}{dx}}$.

Q 8. Find the general solution of $\frac{dy}{dx}=2xe^{x^2-y}$.

Short Answer Type-I Questions

Q 9. Find the particular solution of the differential equation $\frac{dy}{dx}=y\tan x$, when $y(0)=1$.

Q 10. Solve $\frac{dy}{dx}+2xy=y$.

Short Answer Type-II Questions

Q 11. Solve the differential equation:
$x\frac{dy}{dx}=y-x\cos^2\left(\frac{y}{x}\right)$

Q 12. Find the solution of the differential equation:
$\frac{dy}{dx}=\frac{x+y+5}{2(x+y)+3}$.

Long Answer Type Questions

Q 13. Find the equation of the curve, which passes through the point $(1,-1)$ for the differential equation $xy\frac{dy}{dx}=(x+2)(y+2)$.

Q 14. Solve the differential equation $x\frac{dy}{dx}-y=\log x$, given that $y(1)=0$.